

An Analysis of a queue with length distribution according to its priorities

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ABSTRACT : Abstract. We consider a single server multi-class queueing model with Poisson arrivals and relative priorities. For this queue, we derive a system of equations for the transform of the queue length distribution. Using this system of equations we find the moments of the queue length distribution as a solution of linear equations.

length distribution as a solution of linear equations. Besides, numerical examples are given.

Introduction

We consider a multi-class queueing model with relative priorities. In the relative priority service discipline for a single server (processor) system with K classes of customers, if at some service completion there are n_j customers of class $j, j = 1, \dots, K$, then the next customer to commence service is selected from class i customers with probability.

Transform of the queue length distribution

We consider an $M/G/1$ queue with relative priorities and K classes of customers. Each class i customer has a positive priority parameter $p_i, i = 1, \dots, K$. Customers of class i arrive in a Poisson stream with rate λ_i . The overall arrival rate is $\lambda = \sum_{i=1}^K \lambda_i$. The service times of class i customers, denoted by random variable X_i , have an identical distribution function $B_i(t)$ with Laplace-Stieltjes transform $B_i^*(s) = \int_0^\infty e^{-st} dB_i(t)$. The traffic intensity for class i customer is $\rho_i = \lambda_i E[X_i]$ and the total traffic intensity is $\rho = \sum_{j=1}^K \rho_j$.

$$\frac{n_i p_i}{\sum_{j=1}^K n_j p_j}, \quad i = 1, \dots, K.$$

Once a customer has started service, it is served without interruption until completion.

Let $N_i(t), i = 1, \dots, K$, be the number of class i customers in the system at time t . Let τ_n be the n th departure epoch. Then $\{(N_1(\tau_n +), \dots, N_K(\tau_n +)) : n = 1, 2, \dots\}$ is a Markov chain, called an embedded Markov chain (EMC). We observe that

Relative priority model is related to the well-known model of discriminatory processor sharing (DPS), see the recent survey [2]. An essential difference with DPS is that for DPS all customers in the system are served simultaneously by a single processor, whereas in relative priority model, the processor serves customers one at a time until their service has been completed.

A single server multi-class queueing model with relative priorities was first suggested in [4]. For the analysis of queueing model with relative priorities it seems that Haviv and van der Wal [5] is the only known result in open literatures. Haviv and van der Wal [5] obtained the mean waiting times for the $M/G/1$ queue with relative priorities.

$$\begin{aligned} & P((N_1(\tau_{n+1} +), \dots, N_K(\tau_{n+1} +))) \\ &= (1_1, \dots, 1_K) | (N_1(\tau_n +), \dots, N_K(\tau_n +)) = (n_1, \dots, n_K) \\ &= \begin{cases} \sum_{i=1}^K \frac{\lambda_i}{\lambda} b_i(1_1, \dots, 1_K) & \text{if } (n_1, \dots, n_K) = (0, \dots, 0), \\ \sum_{i=1}^K \frac{n_i p_i}{n_1 p_1 + \dots + n_K p_K} b_i((1_1, \dots, 1_K) - (n_1, \dots, n_K) + 1_i) & \text{if } (n_1, \dots, n_K) \neq (0, \dots, 0), \end{cases} \end{aligned} \tag{2.1}$$

In this paper we consider a single server multi-class queueing model with Poisson arrivals and relative priorities. For this queue, we derive a system of equations for the transform of the queue length distribution. Using this system of equations we find the moments of the queue

where

$$\text{if } (\mathbf{l}_1, \dots, \mathbf{l}_K) = (0, \dots, 0),$$

$$b_j(\mathbf{l}_1, \dots, \mathbf{l}_K) = \begin{cases} 0 \\ \int_0^\infty e^{-\lambda t} \frac{\lambda_1^{l_1} \dots \lambda_K^{l_K} t^{l_1 + \dots + l_K} dt}{l_1! \dots l_K!} dB_j(t) \end{cases}$$

otherwise,

which is the probability that the number of each class i customer which arrives during the service time of class j customer, is $l_i, i=1, \dots, K$. Further \mathbf{l}_i denotes a K -dimensional row vector whose i th component is 1 and all other components are 0.

It can be shown that the EMC is positive recurrent if and only if $\rho < 1$. We assume Let be the stationary distribution of the EMC :

$$\pi(\mathbf{l}_1, \dots, \mathbf{l}_K) = \lim_{n \rightarrow \infty} P((N_1(\tau_n +), \dots, N_K(\tau_n +)) = (\mathbf{l}_1, \dots, \mathbf{l}_K))$$

Let us denote by $\Pi(z_1, \dots, z_K)$ the probability generating function of $\pi(\mathbf{l}_1, \dots, \mathbf{l}_K)$:

$$\Pi(z_1, \dots, z_K) \equiv \sum_{l_1=0}^{\infty} \dots \sum_{l_K=0}^{\infty} \pi(\mathbf{l}_1, \dots, \mathbf{l}_K) z_1^{l_1} \dots z_K^{l_K} \quad \text{and} \quad \Pi(\mathbf{l}_1, \dots, \mathbf{l}_K) = (1-\rho) \sum_{i=1}^K \frac{\lambda_i}{\lambda} B_i^* \left(\lambda - \sum_{j=1}^K \lambda_j z_j \right)$$

From (2.1), we have

$$\begin{aligned} & \Pi(z_1, \dots, z_K) \\ & \pi(n_1, \dots, n_K) \\ &= \pi(0, \dots, 0) \sum_{i=1}^K \frac{\lambda_i}{\lambda} B_i^* \left(\lambda - \sum_{j=1}^K \lambda_j z_j \right) + \sum_{(n_1, \dots, n_K) \neq (0, \dots, 0)} \dots \sum \\ & \times \sum_{i=1}^K \frac{n_i p_i}{n_1 p_1 + \dots + n_K p_K} z_1^{n_1} \dots z_i^{n_i-1} \dots z_K^{n_K} B_i^* \left(\lambda - \sum_{j=1}^K \lambda_j z_j \right) \end{aligned} \quad (2.3)$$

Since, equation (2.3) is some what cumbersome to work with directly. We define

$$r(z_1, \dots, z_K) \equiv \sum_{(l_1, \dots, l_K)} \dots \sum_{\neq (0, \dots, 0)} \frac{\pi(\mathbf{l}_1, \dots, \mathbf{l}_K)}{l_1 p_1 + \dots + l_K p_K} z_1^{l_1} \dots z_K^{l_K} \quad (2.4)$$

Further, we not that

$$\pi(0, \dots, 0) = 1 - \rho \quad (2.5)$$

In the following theorem we have a system of equations for the probability generating function of the queue length distribution.

Theorem 1. (a) $\Pi(z_1, \dots, z_K)$ and $r(z_1, \dots, z_K)$ are related by

$$\Pi(z_1, \dots, z_K) = 1 - \rho + \sum_{i=1}^K p_i z_i \frac{\partial}{\partial z_i} r(z_1, \dots, z_K) \quad 2.6$$

(b) $r(z_1, \dots, z_K)$ satisfies

$$\sum_{i=1}^K p_i \left(z_i - B_i^* \left(\lambda - \sum_{j=1}^K \lambda_j z_j \right) \right) \frac{\partial}{\partial z_i} (z_1, \dots, z_K) = (\rho - 1) \left(1 - \sum_{i=1}^K \frac{\lambda_i}{\lambda} B_i^* \left(\lambda - \sum_{j=1}^K \lambda_j z_j \right) \right) \quad (2.7)$$

Proof. Part (a) is immediate from (2.2), (2.4) and (2.5). Now, by using (2.4) and (2.5), we can rewrite (2.3) as

$$+ \sum_{i=1}^K p_i \frac{\partial}{\partial z_i} r(z_1, \dots, z_K) B_i^* \left(\lambda - \sum_{j=1}^K \lambda_j z_j \right)$$

Part (b) follows immediately from substituting (2.6) into the above equation.

Note that the equation (2.7) reduces to

$$\Pi(z_1, \dots, z_K) = \frac{1 - \rho}{1 - \rho z}$$

which is the probability generating function of the queue length in the ordinary M/M/1 queue.

Theorem 1 helps us obtain the moments of the queue length for the M/G/1

queue with relative priorities, as we see in the following section.

Moments of the queue length distribution

In this section we show how to obtain the first and second moments of the queue length. To do this, we define the

following moments:

$$L_{i_1 \dots i_j}^j \equiv \frac{\partial^j}{\partial z_{i_1} \dots \partial z_{i_j}} \Pi(z_1, \dots, z_K) \Big|_{z_1 = \dots = z_K = 1}$$

$$R_{i_1 \dots i_j}^j \equiv \frac{\partial^j}{\partial z_{i_1} \dots \partial z_{i_j}} r(z_1, \dots, z_K) \Big|_{z_1 = \dots = z_K = 1},$$

where $j=1,2,\dots,1 \leq i_l, \leq K$ and $1 \leq l \leq j$. We also define, for $1 \leq l_k \leq j$, $R_{i_1 \dots i_j}^{j-k+1}(i_{l_1}, \dots, i_{l_k})$

$$\equiv \frac{\partial^{j-k+1}}{\partial z_{i_1} \dots \partial z_{i_{l_1-1}} \partial z_{i_{l_1+1}} \dots \partial z_{i_{l_k-1}} \partial z_{i_{l_k+1}} \dots \partial z_{i_j}} r(z_1, \dots, z_K) \Big|_{z_1 = \dots = z_K = 1}$$

Which means that we take partial derivatives with respect to variables $z_{i_1}, \dots, z_{i_j}, z_i$ except $z_{i_{l_1}}, \dots, z_{i_{l_k}}$. For example,

$$R_{i_1 \dots i_j}^j \dots i_j(i_{l_1}) = \frac{\partial^j}{\partial z_{i_1} \dots \partial z_{i_{l_1-1}} \partial z_{i_{l_1+1}} \dots \partial z_{i_j}} r(z_1, \dots, z_K) \Big|_{z_1 = \dots = z_K = 1}$$

More specifically,

$$R_{1236}^3(2) = R_{136}^3, R_{123}^2(2) = R_{13}^2, R_{246}^1(2,4) = R_6^1.$$

We have the following relations on the moments of the queue length.

Theorem 2. For $j=1,2,\dots,1 \leq i_l \leq K$ and $1 \leq l \leq j$,

$$L_{i_1 \dots i_j}^j = \sum_{i=1}^K p_i R_{i_1}^{j+1} \dots i_j + \sum_{i=1}^j p_{i_1} R_{i_1}^j \dots i_j, \quad (3.1)$$

and

$$\begin{aligned} & \sum_{i=1}^j p_{i_1} R_{i_1}^j \dots i_j - \sum_{i=1}^K \sum_{k=1}^j \sum_{1 \leq l_1 < \dots < l_k \leq j} \\ & p_i \phi_{i, i_{l_1} \dots i_{l_k}} \dots i_{l_1} \dots i_{l_k} R_{i_1 \dots i_j}^{j-k+1}(i_{l_1}, \dots, i_{l_k}) \\ & = (1-\rho) \sum_{i=1}^K \frac{\lambda_i}{\lambda} \phi_{i, i_1 \dots i_j}, \end{aligned} \quad 3.2$$

where $\phi_{i, i_1 \dots i_k}$ is given by

$$\phi_{i, i_1 \dots i_k} \equiv \frac{\partial^k}{\partial z_{i_1} \dots \partial z_{i_k}} B_i^* \left(\lambda - \sum_{j=1}^K \partial_j z_j \right) \Big|_{z_1 = \dots = z_K = 1}$$

Proof. The equation (3.1) is derived by taking partial derivatives of (2.6) with respect to $z_{i_1}, z_{i_2}, \dots, z_{i_j}$ and evaluating at $z_1 = \dots = z_K = 1$. Now we derive (3.2). For notational convenience, we set in equation (2.7).

$$I_i(z_1, \dots, z_K) = p_i \left(z_i - B_i^* \left(\lambda - \sum_{j=1}^K \lambda_j z_j \right) \right) \quad (2.7)$$

Note that $I_i(1, \dots, 1) = 0$. By taking partial derivatives of (2.7) with respect to $z_{i_1}, z_{i_2}, \dots, z_{i_j}$ and evaluating at $z_1 = \dots = z_K = 1$, we have

$$\begin{aligned} & \sum_{i=1}^K \sum_{k=1}^j \sum_{1 \leq l_1 < \dots < l_k \leq j} \frac{\partial^k}{\partial z_{i_{l_1}} \dots \partial z_{i_{l_k}}} I_i(z_1, \dots, z_K) \Big|_{z_1 = \dots = z_K = 1} \\ & R_{i_1 \dots i_j}^{j-k+1}(i_{l_1}, \dots, i_{l_k}) \end{aligned} \quad (3.3)$$

$$= (1-\rho) \sum_{i=1}^K \frac{\lambda_i}{\lambda} \phi_{i, i_1 \dots i_j}$$

Observe the for $k=1$, the left hand side of (3.3) becomes

$$\sum_{i=1}^j p_{i_{l_1}} R_{i_{l_1}}^j - \sum_{i=1}^K \sum_{l_1=1}^j p_i \phi_{i, i_{l_1}} R_{i_1 \dots i_j}^j(i_{l_1}),$$

and for $k \geq 2$,

$$\begin{aligned} & \sum_{i=1}^K \sum_{k=2}^j \sum_{1 \leq l_1 < \dots < l_k \leq j} \frac{\partial^k}{\partial z_{i_{l_1}} \dots \partial z_{i_{l_k}}} I_i(z_1, \dots, z_K) \Big|_{z_1 = \dots = z_K = 1} \\ & R_{i_1 \dots i_j}^{j-k+1}(i_{l_1}, \dots, i_{l_k}) \end{aligned}$$

$$= -\sum_{i=1}^K \sum_{k=2}^j \sum_{1 \leq l_1 < \dots < l_k \leq j} p_i \phi_{i, i_1 \dots i_k} R_{i_1 \dots i_j}^{j-k+1} (i_{l_1}, \dots, i_{l_k}) = (1-\rho) \frac{\lambda_k \lambda_l \lambda_m}{\lambda} \sum_{i=1}^K \lambda_i E(X_i^3), \quad l \leq k \leq l \leq m \leq K \quad (3.6)$$

Combining the above two equations, we can rewrite (3.3) an (3.2). This finishes the derivation of (3.2)

Now we describe in detail how to calculate the first and second moments of the queue length. From (3.2) with $j=1$, a system of linear equations for is given as follows :

(3.4)

From (3.2) with $j=2$, a system of linear equations for is $R_k^l, 1 \leq k \leq K$, given as follows :

$$p_k R_k^1 - \lambda_k \sum_{i=1}^K p_i E(X_i) R_i^1 = \frac{\lambda_k}{\lambda} (1-\rho) \rho, \quad k=1, \dots, K. \quad (3.4)$$

From (3.2) with $j=2$, a system of linear equations for $R_{kl}^2, 1 \leq k, l \leq K$, is given as follows :

$$(p_k + p_l) R_{kl}^2 = \lambda_k \sum_{i=1}^K p_i E(X_i) R_{li}^2 - \lambda_l \sum_{i=1}^K p_i E(X_i)$$

$$R_{ki}^2 - \lambda_k \lambda_l \sum_{i=1}^K p_i E(X_i^2) R_i^1$$

$$= (1-\rho) \frac{\lambda_k \lambda_l}{\lambda} \sum_{i=1}^K \lambda_i E(X_i^2), \quad 1 \leq k \leq l \leq K. \quad (3.5)$$

From (3.2) with $j=3$, a system of linear equations for $R_{klm}^3, 1 \leq k, l, m \leq K$, is given as follows :

From (3.2) with $j=3$, a system of linear equations for $R_{klm}^3, 1 \leq k, l, m \leq K$, is given as follows :

$$(p_k + p_l + p_m) R_{klm}^3 - \lambda_k \sum_{i=1}^K p_i E(X_i) R_{lmi}^3 - \lambda_l \sum_{i=1}^K$$

$$p_i E(X_i) R_{kmi}^3$$

$$- \lambda_m \sum_{i=1}^K p_i E(X_i) R_{kli}^3 - \lambda_k \lambda_l \sum_{i=1}^K p_i E(X_i^2) R_{mi}^2 - \lambda_l \lambda_m$$

$$\sum_{i=1}^K p_i E(X_i^2) R_{ki}^2$$

$$- \lambda_m \lambda_k \sum_{i=1}^K p_i E(X_i^2) R_{li}^2 - \lambda_k \lambda_l \lambda_m \sum_{i=1}^K p_i E(X_i^3) R_i^1$$

Thus we can obtain the first and second moments of the queue length as follows. Using (3.4), we first obtain

$R_k^1, k=1, \dots, K$. Secondly, we obtain

$R_{kl}^2, k, l=1, \dots, K$, solving the system of linear equations (3.5). Lastly, we obtain the mean queue length

$L_k^1, k=1, \dots, K$, from

$$L_k^1 = \sum_{i=1}^K p_i R_{ki}^2 + p_k R_k^1 \quad (3.7)$$

by (3.1). Then, solving the system of linear equations (3.6)

yields $R_{klm}^3, k, l, m=1, \dots, K$. Similarly, we obtain the

second moment $L_{kl}^2, k, l=1, \dots, K$, from

$$L_{kl}^2 = \sum_{i=1}^K p_i R_{kli}^3 + p_k R_{kl}^2 + p_l R_{kl}^2 \quad (3.8)$$

by (3.1).

In summary, we have the following procedure for the first and second moments of the queue length.

Procedure for the first and second moments of the queue length

1. Calculate $R_k^1, k=1, \dots, K$, by solving (3.4)

2. Calculate $R_{kl}^2, k, l=1, \dots, K$, by solving (3.5)

3. Calculate the mean queue length $L_k^1, k=1, \dots, K$, by (3.7).

4. Calculate $R_{klm}^3, k, l, m=1, \dots, K$, by solving (3.6)

5. Calculate the second moment of the queue length

$L_{kl}^2, k, l=1, \dots, K$, by (3.8).

Remark. In a similar manner we can obtain the higher moments of the queue length by using the equations (3.1) and (3.2)

APPLICATIONS

In this section we present some numerical examples to compute the mean, variance and squared coefficient of

variation of the queue length. We consider the case of $K=2$ customer classes with priority parameters p_1 and p_2 . We assume equal loads of $\rho_1 = \rho_2 = 0.35$, and hence $\rho = 0.7$.

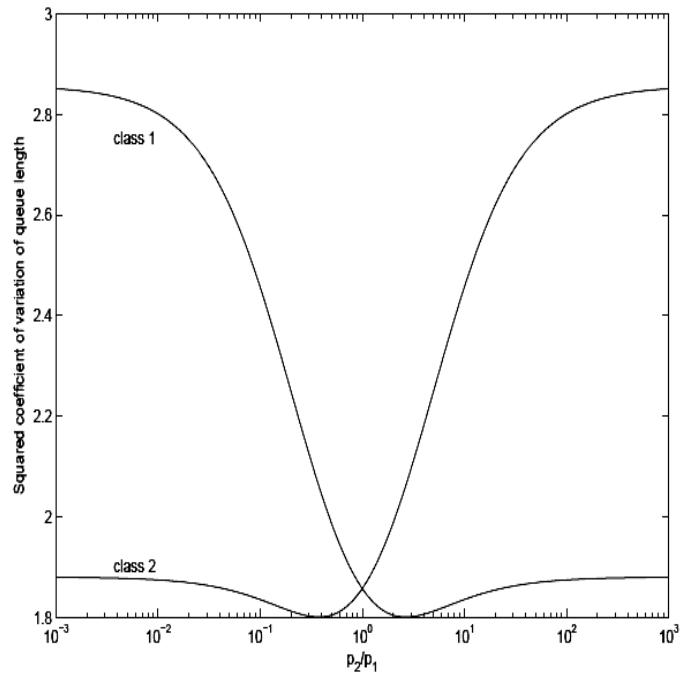


Figure 3. Squared coefficient of variation of queue length

Figure 1. Mean queue length for application-1

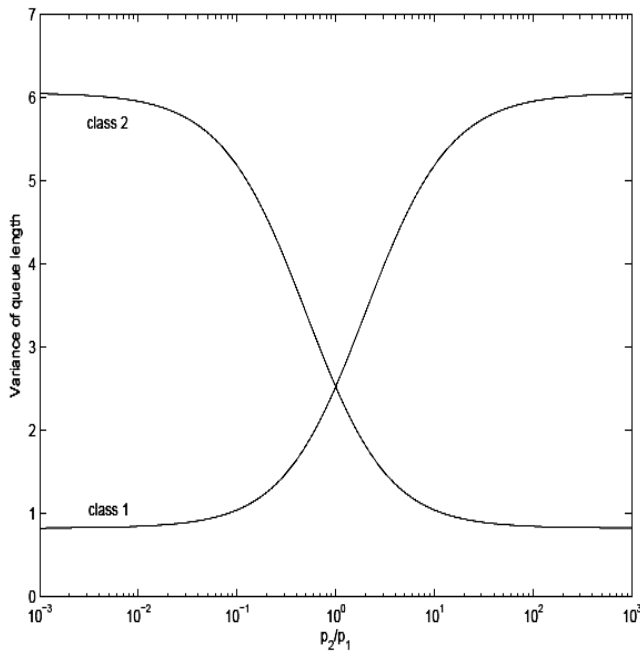


Figure 2. Variance of queue length for application-1

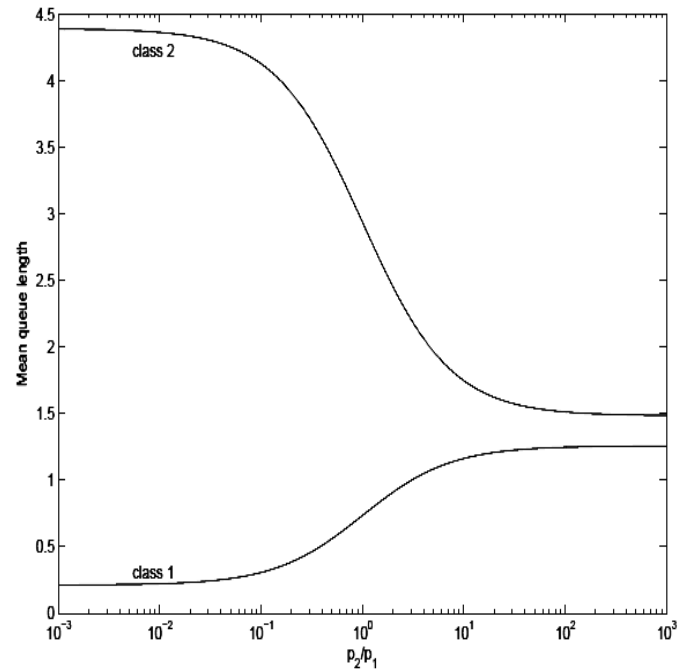


Figure 4. Mean queue length for application-2

density $b(t) = e^{-t}$. Hence $\bar{\epsilon}_1 = \bar{\epsilon}_2 = 0.35$.

Figs. 1-3 show the mean, variance and squared coefficient of variation for the queue length of each class, respectively, varying the priority parameter ratio p_2/p_1 . We observe, as expected, that in the case of $p_2/p_1 = 1$, the mean, variance and squared coefficient of variation for the queue length of class 1 customer equal to those of class 2 customer. It is also observed that the curves in each figure are symmetric with respect to the vertical line passing through $p_2/p_1 = 1$, as expected. From Figs. 1 and 2, we see, as expected, that the mean and variance of the queue length of class 1 customer (resp. class 2 customer) increase (resp. decrease) as the priority parameter ratio p_2/p_1 increases.

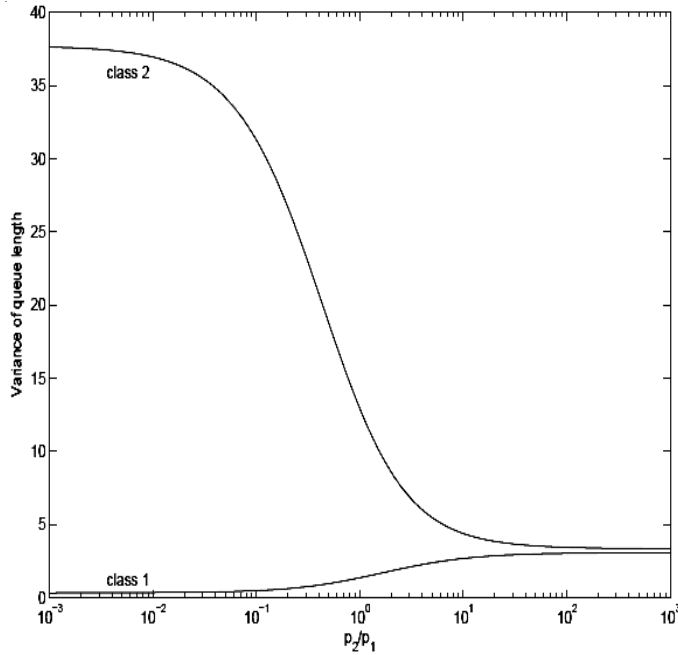


Figure 5 Variance of queue length for application-2

Application 2. We assume that the service times of class 1 and class 2 customers have exponential distributions with densities $b(t) = e^{-t}$ and $b(t) = 4e^{-4t}$, respectively. Hence $\bar{e}_1 = 0.35$ and $\bar{e}_2 = 1.4$. In Figs. 4-6, we plot the mean, variance and squared coefficient of variation for the queue length of each class, respectively, varying the priority parameter ratio p_2/p_1 . As illustrated in Figs. 4 and 5, the mean and variance of the queue length of class 1 customer (resp. class 2 customer) increase (resp. decrease) as the priority parameter ratio p_2/p_1 increases, as we expect. Further, the mean and variance of the queue length of class 2 customer are larger than those of class 1 customer. Fig. 6 illustrates

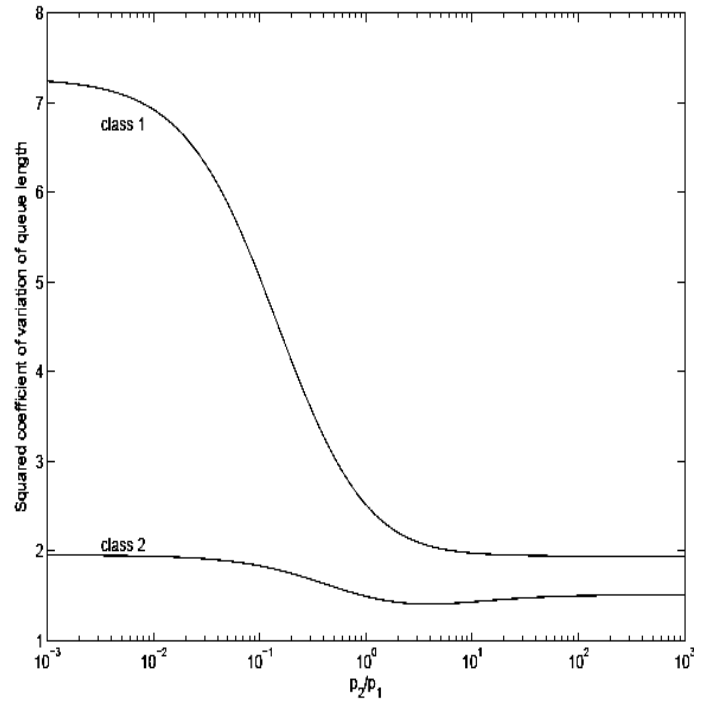


Figure 6 Squared coefficient of variation of queue length for application-2

that the squared coefficient of variation of the queue length of class 2 customer is less affected by the priority parameter ratio p_2/p_1 .

References

1. A. Cobham, Priority Assignment in waiting line problems, *oper. Res.*2 (1954), 70-76.
2. E. Altman, A. Avrachenkov, and U. Ayesta, A survey on discriminatory processor sharing, *Queueing Syst.* 53 (2006), no. 1-2, 53-63.
3. J.H. Hine, I. Mitrani, Complete parameterized families of Job scheduling strategies act. *Inform* 8 (1977) 61-77.
4. M. Haviv, J. Vander Wal, Equilibrium Random strategies for processor sharing and queues with relative priorities, *Probab. Eng. Inform. Sci.*, 11 (1997) 403-412.
5. M. Haviv and J. Vander Wall. Waiting times in queues with relative priorities, *Oper. Res. Lett.* 35 (2007), no. 5, 591-594.