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Common Fixed Point Theorem for New Contractive Condition

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ABSTRACT: We prove a common fixed point theorem for eight mappings using a contractive condition via occasionally weakly biased maps.

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Introduction

Jungck [4] extended commuting and weakly commuting mappings by giving the concept of compatible mappings. Many researchers including Jungck, Murthy and Cho [7], Pathak and Khan [8], Pathak, Cho, Kang and Madharia [9], Ciric, Samet and Vetro [3] gave different types of mapping and proved common fixed point theorem.Jungck and Rhoades[5] introduced the notion of weakly compatible mappings. Jungck and Pathak[6] introduced the concept of biased and weakly biased mappings. Bouhadjera and Djoudi[2] introduced the concept of occasionally weakly biased mappings.

In this paper, we prove a common fixed point theorem for eight mappings using a contractive condition via occasionally weakly biased mapping.

Preliminaries

Difinition 1 : [4] Let(X,d) be a metric space. Self maps f and g are said to be compatible if

$\lim d(fgx_n, gfx_n) = 0$,

whenever $\{ x_n \}$ is a sequence in X such that

$$\lim_{n\to 8} fx_n = \lim_{n\to 8} gx_n = t \text{ for some } t \in X.$$

Difinition 2 : [5] Self mappings f and g of a metric space (X,d) are said to be weakly compatible if they commute at their coincidence points, i.e.ft=gt for some $t \in X$, then fgt = gft.

Difinition 3 : [1] Two self maps f and g of a set X are occasionally weakly compatible iff, there is a point t in X which is a coincidence point of f and g at which f and g commute.

Difinition 4 : [6] The pair {f,g} is g- biased and f-biased, respectively,iff whenever$\{x_n\}$ is a sequence in X and $fx_n, gx_n \to t \in X$,then

$$\alpha\, d(gfx_n, gx_n) \le \alpha\, d(fgx_n, fx_n),$$

$$\alpha\, d(fgx_n, fx_n) \le \alpha\, d(gfx_n, gx_n),$$

respectively, if α = lim inf and if α = lim sup.

The pair {f,g} is compatible then it is both f and g biased. However the converse is not true in general.

Difinition 5 : [6] The pair {f,g} is weakly g- biased and f-biased, respectively, iff fp = gp implies

$$d(gfp, gp) = d(fgp, fp),$$

$$d(fgp, fp) = d(gfp, gp),$$

respectively. Clearly, every biased maps are weakly biased maps respectively,but the converse is false in general.

Difinition 6 : [2] Let f and g be self-maps of a set X.The pair {f,g} is said to be occasionally weakly g-biased and f-biased, respectively, iff there exists a point p in X such that fp = gp implies

$$d(gfp, gp) \le d(fgp, fp),$$

$$d(fgp, fp) \le d(gfp, gp),$$

respectively.Clearly, every f-biased and g-biased maps is occasionally weakly f-biased and occasionally weakly g-biased respectively.

However the converse is not true in general.

To this end consider the following.

Example 1 : Let $X = [0,\infty)$ with the usual metric $d(x, y)=|x-y|$.

Define f, g: X à X by

$$fx = \begin{cases} 9x^2 \text{ if } x \in [0,1] \\ 16/x \text{ if } x \in (1,8) \end{cases} \qquad gx = \begin{cases} 1 \text{ if } x \in [0,1] \\ 4x \text{ if } x \in (1,8) \end{cases}$$

we have fx = gx iff x = 1/3 or x = 2 and

$6 = d(fg(2), f(2)) \leq d(gf(2), g(2)) = 24$

i.e. the pair {f, g} is occasionally weakly f- biased, but

$8 = d(fg(1/3), f(1/3)) \not\leq d(gf(1/3), g(1/3)) = 0$

i.e. the pair {f,g} is not weakly f- biased.

On the other hand we have

$0 = d(gf(1/3), g(1/3) \leq d(fg(1/3), f(1/3)) = 8$

i.e. the pair {f, g} is occasionally weakly g-biased, but as

$24 = d(gf(2),g(2)) \not\leq d(fg(2),f(2)) = 6$

then f and g are not weakly g- biased.

Remark 1 : Note that from the preceding example we have

$fg(2) = 2 \neq 32 = gf(2)$

and $fg(1/3) = 9 \neq 1 = gf(1/3)$

i.e. f and g are not occasionally weakly compatible maps. On the other hand it is clear from the definitions that if f and g are occasionally weakly compatible maps then f, g are both occasionally weakly f-biased and g-biased.

Therefore, occasionally weakly compatible map is a subclass of occasionally weakly biased maps.

Main Result

Theorem 1 : Let f, g, h, k, t, u, v, w be self mappings of a metric space (X, d) such that

1.1 f(X)? w(X), g(X)? v(X), h(X)? u(X), k(X)? t(X);

1.2 (a) $d(fx, gy) = \alpha\, d(tx,uy) + \beta[d(fx,tx)+d(gy, uy)] + \gamma[d(tx,gy)+d(uy,fx)]$,

(b) $d(gx, hy) = \alpha\, d(ux,vy) + \beta[d(gx,ux)+d(hy,vy)] + \gamma[d(ux,hy)+d(vy,gx)]$,

(c) $d(hx, ky) = \alpha\, d(vx,wy) + \beta[d(hx,vx)+d(ky,wy)] + \gamma[d(vx,ky)+d(wy,hx)]$,

(d) $d(kx, fy) = \alpha\, d(wx,ty) + \beta[d(kx,wx)+d(fy, ty)] + \gamma[d(wx,fy)+d(ty,kx)]$,

for all x, y $\in$ X and α, β, and γ be non – negative reals such that

$\alpha+2\beta+2\gamma<1$;

1.3 (f,t),(g,u),(h,v),(k,w) are occasionally weakly t-biased, occasionally weakly u-biased, occasionally weakly v-biased and occasionally weakly w- biased maps respectively.

Then f,g,h,k,t,u,v,w have a unique common fixed point.

Proof : Since pairs of mappings (f,t),(g,u),(h,v),(k,w) are occasionally weakly t-biased, occasionally weakly u-biased, occasionally weakly v-biased and occasionally weakly w-biased maps respectively, then there exists four points a, b, c and d are in X s.t.

fa = ta implies $d(tfa, ta) \leq d(fta, fa)$,

gb = ub implies $d(ugb, ub) \leq d(gub,gb)$,

hc = vc implies $d(vhc, vc) \leq d(hvc, hc)$,

kd = wd implies $d(wkd, wd) \leq d(kwd, kd)$.

First, we will prove that fa = gb = hc = kd.

First, we prove that fa = gb. From inequality 1.2(a), we have

$$d(fa,gb) = \alpha\, d(ta, ub) + \beta[d(fa,ta) + d(gb,ub)] + \gamma[d(ta,gb) + d(ub,fa)]$$
$$= \alpha\, d(fa,gb) + \beta[d(fa,fa) + d(gb,gb)] + \gamma[d(fa,gb) + d(gb,fa)]$$
$$= (\alpha+2\gamma)\, d(fa,gb)$$
$$= d(fa,gb), \text{ a contradiction.}$$

Therefore, fa = gb.

Now, we prove that gb = hc. From inequality 1.2 (b), we have

$$d(gb,hc) = \alpha\, d(ub,vc) + \beta[d(gb,ub) + d(hc,vc)] + \gamma[d(ub,hc) + d(vc,gb)]$$
$$= \alpha\, d(gb,hc) + \beta[d(ub,ub) + d(vc,vc)] + \gamma[d(gb,hc) + d(hc,gb)]$$
$$= (\alpha+2\gamma)\, d(gb,hc)$$
$$= d(gb,hc), \text{ a contradiction.}$$

Therefore, gb = hc.

Now, we prove that hc = kd. From inequality 1.2(c), we have

$$d(hc,kd) = \alpha\, d(vc,wd) + \beta[d(hc,vc) + d(kd,wd)] + \gamma[d(vc,kd) + d(wd,hc)]$$
$$= \alpha\, d(hc,kd) + \beta[d(hc,hc) + d(kd,kd)] + \gamma[d(hc,kd) + d(kd,hc)]$$
$$= (\alpha+2\gamma)\, d(hc,kd)$$
$$= d(hc,kd), \text{ a contradiction.}$$

Therefore, hc = kd.

Now, we prove that kd = fa. From inequality 1.2(d), we have

$$d(kd,fa) = \alpha d(wd,ta) + \beta[d(kd,wd) + d(fa,ta)] + \gamma[d(wd,fa) + d(ta,kd)] \quad 4$$
$$= \alpha d(kd,fa) + \beta[d(kd,kd) + d(fa,fa)] + \gamma[d(kd,fa) + d(fa,kd)]$$
$$= (\alpha+2\gamma)\, d(kd,fa)$$
$$= d(kd,fa), \text{ a contradiction.}$$

Therefore kd = fa.

Hence fa = gb = hc = kd.

Now occasionally weakly t-biased of (f,t) implies d(tfa,ta) = d(fta, fa),

occasionally weakly u-biased of (g,u) implies d(ugb,ub) = d(gub,gb),

occasionally weakly v-biased of (h,v) implies d(vhc,vc) = d(hvc,hc),

occasionally weakly w-biased of (k,w) implies d(wkd,wd) = d(kwd,kd).

On the other hand, we obtain

fa = ta implies ffa = fta, tfa = tta,

ub = gb implies gub = ggb, uub = ugb,

vc = hc implies hvc = hhc, vvc = vhc and

wd = kd implies kwd = kkd, wwd = wkd.

Now, we show that fa is a common fixed point of f,g,h,k,t,u,v,w using 1.2(a), we obtain

$$\begin{aligned} d(ffa,fa) = d(ffa,gb) &= \alpha\, d(tfa,ub) + \beta[d(ffa,tfa)+d(gb,ub)]+\gamma[d(tfa,gb)+d(ub,ffa)] \\ &= \alpha\, d(tfa,ta) + \beta[d(ffa,ta)+d(ta,tfa)+0]+ \gamma[d(tfa,ta)+d(fa,ffa)] \\ &= \alpha\, d(fta,fa)+ \beta[d(ffa,fa)+d(fta,fa)]+ \gamma[d(fta,fa)+d(ffa,fa)] \\ &= \alpha\, d(ffa,fa)+ \beta[d(ffa,fa)+d(ffa,fa)]+ \gamma[d(ffa,fa)+d(ffa,fa)] \\ &= (\alpha+2\beta+2\gamma)\, d(ffa,fa) \\ &< d(ffa,fa), \text{ a contradiction.} \end{aligned}$$

Therefore, ffa = fa.

Hence, fa is a fixed point of f. Further, it is easy to show that fa is also a fixed point of t. Thus, fa is a common fixed point of {f, t}. Similarly, one can show that gb is a common fixed point of {g,u}, hc and kd is a common fixed point of {h,v} and {k,w} respectively. Since fa = gb = hc = kd we therefore conclude that fa is a common fixed point of f, g, h, k, t, u, v, w.The proof is similar.

Now check the uniqueness of common fixed point. Put fa=r therefore r is a common fixed point of mapping f, g, h, k, t, u, v, w. Now, let r and s be two different common fixed point of mappings f, g, h, k, t, u, v, w i.e.

fr = gr = hr = kr = tr = ur = vr = wr = r and fs = gs = hs = ks = ts = us = vs = ws = s 5

$$\begin{aligned} d(r,s) &= d(fr,gs) \\ &= \alpha\, d(tr,us) + \beta\, [d(fr,tr) + d(gs,us)] + \gamma\, [d(tr,gs) + d(us,fr)] \\ &= \alpha\, [d(r,s)] + \beta\, [d(r,r) + d(s,s)] + \gamma\, [d(r,s) + d(s,r)] \\ &= (\alpha + 2\gamma)\, d(r,s) \\ &= d(r,s), \text{ a contradiction.} \end{aligned}$$

Therefore, r = s.

Hence fa is a common fixed point of f, g, h, k, t, u, v, w.

This completes the proof of the theorem.

Now, we give an example to illustrate the above theorem.

Example 2 : Let $X = [0,\infty)$ with usual metric $d(x,y) = |x-y|$.

Define f = g = h = k, t, u, v, w : $X \to X$ by

$$f(x) = g(x) = h(x) = k(x) = \begin{cases} 1 \text{ if } x \in [0,1] \\ 5x \text{ if } x \in (1,8) \end{cases}$$

$$t(x) = \begin{cases} 3x^2 \text{ if } x \in [0,1] \\ 25/x \text{ if } x \in (1,8) \end{cases} \qquad u(x) = \begin{cases} 6x^2 \text{ if } x \in [0,1] \\ 35/x \text{ if } x \in (1,8) \end{cases}$$

$$v(x) = \begin{cases} 8x^2 \text{ if } x \in [0,1] \\ 55/x \text{ if } x \in (1,8) \end{cases} \qquad w(x) = \begin{cases} 9x^2 \text{ if } x \in [0,1] \\ 75/x \text{ if } x \in (1,8) \end{cases}$$

Clearly, 1.1 f(X)? w(X), g(X)? v(X), h(X)? u(X), k(X)? t(X);

1.2 (a),(b),(c),(d) is true for all x,y Є X;

Now we shall show that (f,t) is occasionally weakly t-biased,

we have f(x) = t(x) iff $x = 1/\sqrt{3}$ or $x = \sqrt{5}$

$4\sqrt{5} = d(tf\sqrt{5}, t\sqrt{5}) = d(ft\sqrt{5}, f\sqrt{5}) = 20\sqrt{5}$

$2 = d(tf1/\sqrt{3}, t1/\sqrt{3}) \neq d(ft1/\sqrt{3}, f1/\sqrt{3}) = 0$

i.e. the pair (f,t) is occasionally weakly t-biased,

but $ft\sqrt{5} = 125/\sqrt{5}$? $5/\sqrt{5} = tf\sqrt{5}$

and $ft1/\sqrt{3} = 1$? $3 = tf1/\sqrt{3}$

i.e. f and t are not occasionally weakly compatible maps.

similarly (g,u) is occasionally weakly u-biased,

we have g(x) = u(x) iff $x = 1/\sqrt{6}$ or $x = \sqrt{7}$

$4\sqrt{7} = d(ug\sqrt{7}, u\sqrt{7}) = d(gu\sqrt{7}, g\sqrt{7}) = 20\sqrt{7}$

$5 = d(ug1/\sqrt{6}, u1/\sqrt{6}) \neq d(gu1/\sqrt{6}, g1/\sqrt{6}) = 0$

i.e. the pair (g,u) is occasionally weakly u-biased,

but $gu\sqrt{7} = 175/\sqrt{7}$? $7/\sqrt{7} = ug\sqrt{7}$

and $gu1/\sqrt{6} = 1$? $6 = ug1/\sqrt{6}$

i.e. g and u are not occasionally weakly compatible maps.

similarly (h,v) is occasionally weakly v-biased,

we have h(x) = v(x) iff $x = 1/\sqrt{8}$ or $x = \sqrt{11}$

$4\sqrt{11} = d(vh\sqrt{11}, v\sqrt{11}) = d(hv\sqrt{11}, h\sqrt{11}) = 20\sqrt{11}$

$7 = d(vh1/\sqrt{8}, v1/\sqrt{8}) \neq d(hv1/\sqrt{8}, h1/\sqrt{8}) = 0$

i.e. the pair (h,v) is occasionally weakly v-biased,

but $hv\sqrt{11} = 275/\sqrt{11}$? $11/\sqrt{11} = vh\sqrt{11}$

and $hv1/\sqrt{8} = 1$? $8 = vh1/\sqrt{8}$

i.e. h and v are not occasionally weakly compatible maps.

similarly (k,w) is occasionally weakly w-biased,

we have $k(x) = w(x)$ iff $x = 1/3$ or $x = \sqrt{15}$

$4\sqrt{15} = d(wk\sqrt{15}, w\sqrt{15}) = d(kw\sqrt{15}, k\sqrt{15}) = 20\sqrt{15}$

$8 = d(wk1/3, w1/3) \neq d(kw1/3, k1/3) = 0$

i.e. the pair (k,w) is occasionally weakly w-biased,

but $kw\sqrt{15} = 375/\sqrt{15}$? $15/\sqrt{15} = wk\sqrt{15}$

and $kw1/3 = 1$? $9 = wk1/3$

i.e. k and w are not occasionally weakly compatible maps.

Then f,g,h,k,t,u,v and w satisfy all the conditions of the above Theorem 1 and have a unique common fixed point at x = 1.

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