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GENERALIZATIONS OF q-KONHAUSER POLYNOMIALS

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ABSTRACT

The present paper deals with generalization of q-konhauser polynomials & their properties.

Series Expansion, some important theorem results and Rodrigue's formula.

(1) INTRODUCTION

In this paper, we shall study the generalizations of q-Konhauser polynomials by Al-Salam, W.A. and Verma, A. [1] and q- analogues of polynomials discussed by Raizada [6] . For this purpose we shall define the q-analogues.

(2) PRELIMINARY RESULTS

In this section we shall use the following notations and standard results:

(2.1) $$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i), \quad \text{for} \quad |q|<1$$

(2.2) $$(a;q)_n = (a;q)_\infty / (aq^i;q)_\infty$$

for arbitrary complex number n.

In particular for n = 1, 2, ……………, we have

(2.3) $$(a;q)_n = (1-a)(1-aq)................(1-aq^{n-1})$$

The q- derivative with base q is defined as [1] :

(2.4) $$D_q f(x) = [(f(x) - f(xq)]/x$$

and its n-th iterate as follows:

(2.5) $$D_q^n f(x) = x^{-n} \sum_{j=0}^{n} \frac{[q^{-n}]_j q^j}{[q]_j} f(xq^j)$$

The q- Gamma function may be defined [3] by

(2.6) $$\Gamma q(x) = \frac{[q]_\infty}{[q^x]_\infty}(1-q)^{1-x}, \quad 0 < q < 1.$$

The q- Binomial theorem is given by

(2.7) $$\sum_{n=0}^{\infty}\frac{[a]_n}{[q]_n}x^n = \frac{[ax]_\infty}{[x]_\infty}, \quad |x|<1$$

(2.8) $$(x-y\,;q)_n = [x-y]_n$$
$$= (x-y)(x-qy)\ \ldots\ldots\ldots\ldots\ldots\ (x-q^{n-1}y)$$
$$= x^n[1-y/x]^n$$
$$= x^n\sum_{i=0}^{n}[{n \atop i};q]\,q^{i(i-1)/2}(y/x)^i$$

where

(2.9) $$[{n \atop i};q] = \frac{[q]_n}{[q]_i[q]_{n-i}}$$

The Jackson's [6], q-analogues of Taylor's theorem for polynomials of degree $\leq$ n as follows:

(2.10) $$f(x)=\sum_{r=0}^{n}\frac{x^r[1/x]_r}{[q]_r}[D_q f(x)]_{x=1}$$

The q-analogue of exponential expansion as follows:

(2.11) $$\frac{1}{(a;q)_\infty}=\sum_{n=0}^{\infty}\frac{a^n}{[q]_n}$$

(2.12) $$\int_0^\infty \frac{x^\alpha}{[-px^r]_\infty}dx=\frac{\Gamma[1+\alpha)/r]\Gamma[1-(1+\alpha)/r]}{\Gamma q^r[1-(1+\alpha)/r]}(1-q^r)^{(1+\alpha)/r}$$

The q-analogue of a result of Carlitz [4] with base q^r as follows:

(2.13) $$(q^{-km};q^k)_m=\sum_{i=0}^{m}\frac{(q^{(1+\alpha+km)};q^r)_i}{(q^r;q^r)_i}\,q^{-i(1+\alpha+km)}$$
$$=\sum_{j=0}^{i}\frac{(q^{-ri};q^r)_j}{(q^r;q^r)_j}q^{rj}(q^{-i(1+\alpha+km)};q^k)_n$$

this result is q-analogue of

$$(-m)_n=\sum_{p=0}^{n}[(1+\alpha+km)/r]^p(1/p!)\sum_{j=0}^{p}(-1)^j({p \atop j})(\{1+\alpha+rj\}/k)_n$$

The q-analogue of Jacksons with q^r of Taylor's theorem as follows:

(2.14) $$f(x)=\sum_{i=0}^{m}\frac{x^i(1/x\,;q^r)_i}{(q^r;q^r)_i}[D_q r\,f(x)]_{x=1}$$
$$=\sum_{i=0}^{m}\frac{x^i(1/x;q^r)_i}{(q^r;q^r)_i}.\sum_{i=0}^{j}\frac{(q^{-ri};q^r)_j}{(q^r;q^r)_j}q^{rj}[f(xq^{rj}]_{x=1}$$

now, we take $f(x)=(xq^{1+\alpha};q^k)_m$, we get :

(2.15) $$(xq^{1+\alpha};q^k)_m$$

$$= \sum_{i=0}^{m} \frac{x^i (1/x; q^r)_i}{(q^r; q^r)_i} . \sum_{i=0}^{j} \frac{(q^{-ri}; q^r)_j}{(q^r; q^r)_j} q^{rj} (q^{1+\alpha+rj}; q^k)_m$$

when $x = q^{-1-\alpha-km}$, (2.15) reduces to (3.13)

(3) GENERALIZED q-KONHAUSER POLYNOMIALS

We define below a pair of biorthogonal sets of generalized q-Konhauser polynomials $W_n^{(\alpha)}(x; r, p, k, s, t/q)$ and $U_n^{(\alpha)}(x; r, p, k, s, t/q)$, where $W_n^{(\alpha)}(x; r, p, k, s, t/q)$ is polynomials of degree n and in x^s and $U_n^{(\alpha)}(x; r, p, k, s, t/q)$ is a polynomial of degree n in x^r (r is integer) and for n = 0, 1, 2, ….;

(3.1) $$W_n^{(\alpha)}(x; r, p, k, s, t/q) = \frac{(q^{K+\alpha}; q^\alpha)_{nt}}{p^{tn}(q^t; q^t)_n} \sum_{m=0}^{n} \frac{(q^{-nt}; q^t)_m}{(q^t, q^t)_m} \frac{q^{[tm(tm-1)+tm(\alpha+k-r+i+n]/2} \quad p^{tm} x^{5m}}{(q^{k+\alpha}; q^r)_{tm}}$$

and

(3.2) $$U_n^{(\alpha)}(x; r, p, k, s, t/q) = \frac{1}{(q^r; q^r)_n} \sum_{i=0}^{n} \frac{p^i x^{ri} q^{[ri(ri-1)]/2}}{(q^r; q^r)_i} * \sum_{j=0}^{i} \frac{(q^{-ri}; q^r)_j (q^{k+\alpha+rj)}; q^s)_n s^{rj}}{(q^r; q^r)_j}$$

where s/r = t is a positive integer and (α +k)/r > 0.

BIORTHOGONAL RELATION

The q-polynomial sets $\mathrm{W}_n^{(\alpha)}(\mathrm{x}; \mathrm{r}, \mathrm{p}, \mathrm{k}, \mathrm{s}, \mathrm{t}/\mathrm{q})$ and $\mathrm{U}_n^{(\alpha)}(\mathrm{x}; \mathrm{r}, \mathrm{p}, \mathrm{k}, \mathrm{s}, \mathrm{t}/\mathrm{q})$ are bi orthogonal over the interval (0, ∞).

The biorthogonal relation is given by

(3.3) $$\int_0^\infty W_n^{(\alpha)}(x; r, p, k, s, t/q)\, U_n^{(\alpha)}(x; r, p, k, s, t/q) d\Omega(\alpha, x, r, p, k)$$
$$= \frac{(q^{k+\alpha}; q^r)tn\,(q^s; q^s)n\; q^{-ns+(1-r)nt(t+1)}}{[r;q](q^t; q^t)_n (q^r; q^r)_m P^{(k+\alpha+sm)/x}} \delta_{mn}$$

where dΩ (α,x,r,p) is a continuous distribution function is given by

(3.4) $$d\Omega(\alpha, x, r, p, k) = \frac{Ax^\alpha}{[-px^r]_\infty} dx$$

and

(3.5) $$A = \frac{\Gamma q^r \{1-(k+\alpha)/r]}{\Gamma[1-(k+\alpha)/r]\Gamma[(k+\alpha)/r](1-q^r)^{(k+\alpha)/x}}$$

the nth moment of the distribution is given by

(3.6) $\mu_n = \int_0^\infty x^n dx$

$$= \frac{(q^{k+\alpha};q^r)_j q^{-j(2\alpha+2k-r+n)/2}}{[r;q][p^{(k+\alpha+n)/r}}$$

where j = n/r, a positive integer.
The polynomials (3.1) and (3.2) reduce to q-Konhauser

Polynomials $Z_n^{(\alpha)}(x;k/q) = \frac{[{}_q^{1+\alpha}]_{nk}}{({}_q^k;{}_q^k)} \sum_{j=0}^{n} \frac{(q^{nk};q^k)_{jq}^{(1/2)kj(kj-1)+kj(n+\alpha+1)}}{({}_{q;}^{k}\;{}_{q}^{k})_j \;(q^{1+\alpha})_{kj}} x^{kj}$

$$Y_n^{(\alpha)}(x;k/q) = \frac{1}{[q]_n} \sum_{r=0}^{n} \frac{x^r q^{(1/2)r(r-1)}}{[q]_r} \sum_{j=0}^{r} \frac{[q^{-r}]_j (q^{1+\alpha+j} q^k)_n}{[q]_r} q^j$$

respectively for r = p = k = 1 and s/r = t = k and also subsequently reduces to q-Laguerre

polynomials $L_n^{(\alpha)}(x/q) = \frac{[q^{1+\alpha}]_n}{[q]_n} \sum_{j=0}^{n} \frac{[q^{-n}] q^{1/2j(j+k)+j(\alpha+n)}}{[q]_j [q^{k+\alpha}]_j} x^j$ for k = 1.

To prove biorthogonal relation (3.3) it is necessary and sufficient to show that

(3.7) $I_{n,l} = \int_0^\infty x^{nl} W_n^{(\alpha)}(x;r,p,k,s,t/q) d\Omega(\alpha,x,r,p,k)$ =0 ,0≤ l<n

≠0 l=n

and

(3.8) $I_{n,l} = \int_0^\infty x^{sm} U_n^{(\alpha)}(x;r,p,k,s,t/q) d\Omega(\alpha,x,r,p,k)$ [= 0,0≤ m < n and ≠ 0m = n.

PROOF OF (3.7)

From (3.1) and (3.7), we get :

$$I_{n,l} = \frac{(q^{k+\alpha};q^r)_{nt}}{p^{tn}(q^t;q^t)_n} \sum_{m=0}^{n} \frac{(q^{-nt};q^t)_m}{(q^t;q^t)_m} p^{tm}$$

$$* \frac{q^{[tm(tm-1)+tm(\alpha+k-r+1+n)]/2}}{(q^{k+\alpha};q^r)_{tm}} \int_0^\infty X^{sm+rl} d\Omega(\alpha,x,r,p,k)$$

$$= \frac{(q^{k+\alpha};q^r)_{nt} \quad q^{-1(2\alpha+2k-r+1r)/2}}{[r;q] \quad (q^t;q^t)_n p^{(k+\alpha)/r+1+m}} \sum_{m=0}^{n} \frac{(q^{-nt},q^t)_m}{(q^t;q^t)_m}$$

$$* \; q^{tm(n-1)} (q^{tm+\alpha+k;} q^r)_j$$

$$= \frac{(-1)^1 (q^{k+\alpha};q^r)_{nt}}{[r;q](q^t;q^t) \quad p^{(k+\alpha)/r+q+tm}} \sum_{m=0}^{n} \frac{(u^{-nt};u^t)_m}{(u^t;u^t)_m}$$

$$* \; (u^{tm+\alpha+k};u^r)_1 \quad u^{tm}$$

where u = I/q.
thus, we get :

(3.9) $$I_{n,1}=\frac{(-1)^{1}(q^{k+\alpha};q^{r})_{nt}}{[r;q]\quad(q^{t};q^{t})_{n}\;p^{(k+\alpha)/r+1+tm}}[D_{n}^{n\,t}(xu^{k+\alpha};u^{r})_{1}]_{x=1}$$

Now (x $u^{k+\alpha}$; u^{r}) is a polynomial in of degree 1.
Hence for s = 0, 1,2, 3,……….., n-1. Its nth difference is zero
i.e. $I_{n,1}=0,\ 0\le 1<n.$
For 1 = n, (3.9) becomes

(3.10) $$I_{n,n}=\frac{(-1)^{1}(q^{k+\alpha};q^{r})_{nt}}{[r;q]\quad(q^{t};q^{t})_{n}\;p^{(k+\alpha+n(t+1)/r}}[D_{n}^{n\,t}(xu^{k+\alpha};u^{r})_{n}]_{x=1}$$

The nth derivation formula with base u^{t} is given by

(3.11) $$[D_{n}{}^{t}(xu^{k+\alpha};u^{r})_{n}]=-(q^{t};q^{t})_{n}\,q^{[-nt(n+1)-2(\alpha+k)-r(1+n)]/2}$$

Using (3.11) , (3.10) becomes;

(3.12) $$I_{n,n}=\frac{(-1)^{n}(q^{k+\alpha};q^{r})_{nt}}{[r;q]\quad p^{[(k+\alpha)/r]+(t-1)}}\,q^{[-nt(n+1)-2(\alpha+k)-r(1+n)]/2}\neq 0.$$

This completes proof of (3.7).

Similarly we can prove (3.8)

(4) PROPERTIES OF GENERALIZED q-KONHAUSER POLYNOMIALS:

In this section we give below some of the interesting properties of the polynomials $W_{n}^{(\alpha)}(x,r,p,k,s,t/q)$ and $U_{n}^{(\alpha)}(x,r,p,k,s,t/q)$ which includes generating relations, connection coefficient formula, series expansions, difference recurrence relations and Rodrigues type formula.
The above polynomials possesses the following properties.

(4.1) $$\sum_{n=0}^{\infty}\frac{W_{n}^{(\alpha)}(x,r,p,k,s,t/q)\quad 1^{n}}{(q^{k+\alpha};q^{r})_{tm}}=\frac{f(1x^{s})}{(1p^{t};p^{t})_{\infty}}$$

where $$f(u)=\sum_{m=0}^{\infty}\frac{q^{[tm(tm+2(\alpha+k-r)+m]/2}}{(q^{t};q^{t})_{m}(q^{k+\alpha};q^{r})_{tm}}\;;$$

(4.2) $$W_{n}^{(\alpha)}(xy,r,p,k,s,t/q)$$

$$\sum_{m=0}^{n}\frac{(q^{k+\alpha};q^{r})_{tm}P^{-tm}\quad W_{n-m}^{(\alpha)}(x,r,p,k,s,t/q)}{(q^{k+\alpha};q^{r})_{t(n-m)}\quad(q^{t};q^{t})_{m}}$$

(4.3) If $W_{n}^{(\alpha)}(x,r,p,k,s,t/q)=\sum_{i=0}^{n}C(n,i)W_{i}^{(\beta)}(x;r,p,k,s,t/q)$, where

$$C(n,i)=\frac{(q^{k+\alpha};q^{r})_{tn}\,q^{-ti(\alpha+k+1-\beta)}\,p^{t(i-n)}(q^{t};q^{t})_{i}}{(q^{k+\alpha};q^{r})_{ti}(q^{s};q^{s})_{i}(q^{t};q^{t})_{n-i}}$$

$$*\quad\sum_{j=0}^{n-i}\frac{(q^{-nt+ti};q^{t})_{j}\;(q^{k+\beta+si)};q^{r})_{tj}}{(q^{t};q^{t})_{j}}$$

$$* \quad \frac{(q^{-s(i+j)};q^{s})\, q^{tj(n-ri+\alpha+\beta)+tj[2(1-r)(tj+2ti+1]}}{(q^{t(1+j)};q^{t})_{j}}$$

for r=p= k = p = s = t, this reduces to the connection coefficient for q-Laguerre polynomials.

(4.4) $$D^{1}{}_{u}{}^{r}[x^{k+\alpha+(1-r)(1-t)}D_{u}] \quad W_{n}^{(\alpha)}(x^{\frac{1}{4}};r,p,k,s,t/q)$$

$$= \frac{(-1)^{t}(q^{k+\alpha};q^{r})_{tn}\, x^{\alpha+(1-r)(t-1)}}{(q^{k+\alpha};q^{r})_{t(n-1)} \quad q^{s(t-1)/2}}$$

$$* \quad q^{t(t-k-2r)/2} \quad W_{n-1}^{(\alpha)}(x;r,p,k,s,t/q)$$

(4.5) $$(q^{k+\alpha+sn};q^{r})_{t}W_{n}^{(\alpha)}(x;r,p,k,s,t/q) - p^{t}W_{n+1}^{(\alpha)}(x;r,p,k,s,t/q)$$

$$= p^{t}x^{s} \quad q^{[t(t+2\alpha+2k+1-2r+2n]} * \quad W_{n}^{(\alpha+k)}(xq^{(1-r)t};r,p,k,s,t/q)$$

If $x^{sn} = \sum_{i=0}^{n} D(n,i)W_{i}^{(\alpha)}(x;r,p,k,s,t/q)$

then

(4.6) $$D(n,i) = \frac{(q^{k+\alpha};q^{r})_{tn}\, p^{i(1-n)}(q^{t}\;;q^{t})_{i}(q^{-sn};q^{s})_{i}}{(q^{k+\alpha};q^{r})_{ti} \quad (q^{s};q^{s})_{i}}$$

$$* \quad q^{si-tn[2\alpha+2k-r+n(t+1-r)]/2}$$

(4.7) $$\sum_{n=0}^{\infty}\frac{(q^{r};q^{r})_{n} \qquad t^{n}}{(q^{s};q^{s})_{n}}U_{n}^{(\alpha)}(x;r,p,k,s,t/q)$$

$$= \frac{1}{[1\,;q^{s}]_{\infty}} \; f^{(\alpha,r,p,k)}(1,x,s,t/q)$$

where $f^{(\alpha,r,p,k)}(1,x,s,t/q) = \sum_{j=0}^{n}\frac{q^{sj(sj-1)}(px^{r};q^{r})_{j}(1q^{k+\alpha})^{j}}{(q^{s}\;;q^{s})_{j}}$

(4.8) $$U_{n}^{(\alpha)}(x;r,p,k,s,t/q) = \sum_{m=0}^{n}\frac{(q^{s};q^{s})_{n}(q^{r};q^{r})_{m}(q^{\alpha-\beta};q^{s})_{n-m}}{(q^{s};q^{s})_{m}\;(q^{r};q^{r})_{n}(q^{s};q^{s})_{n-m}}$$

$$* \quad q^{m(\alpha-\beta)}\; U_{m}^{(\beta)}(x;r,p,k,s,t/q)$$

If $x^{rn} = \sum_{i=0}^{n} D(n,i)U_{i}^{(\alpha)}(x;r,p,k,s,t/q)$

then

(4.9) $$D(n,i) = \frac{(q^{-nr};q^{r})_{i}\;(q^{1}\;;q^{1})_{i}}{(q^{k+\alpha};q^{r})_{ti} \quad (q^{s};q^{s})_{i}}\sum_{m=0}^{i}\frac{(q^{-r(n-i)};q^{r})_{m}}{(q^{r(1+i)};q^{r})_{m}}$$

$$* \quad \frac{(q^{t(1+i)};q^{t})_{m}}{(q^{t};q^{t})_{m}}q^{-m(ti+\alpha)+(r-1)m} \quad * \quad (-1)^{n+i}\; q^{-i\alpha-[1i(i+1)]/2+ik+(r-1)i[ti(t+1)+1]/2}$$

(4.10) $$U_{n}^{(\alpha)}(x;r,p,k,s,t/q) = \frac{x^{s-\alpha-k}(-px^{r};q^{r})_{\infty}}{(q^{r};q^{r})_{n}}[D^{ns}{}_{q}\frac{x^{\beta+n}}{(-px^{r/s};q^{r})_{\infty}}]$$

where B = (k + α – s)/s

The properties (4.8), (4.9) and (4.10) reduce for r = p = k = 1, s/r = t = k to thus, we get the relation:

$$C(n,i)=\frac{(q^{k+\alpha};q^{r})_{tn}\quad q^{-ti(\alpha+k+1-\beta)}P^{t(i-n)}(q^{t};q^{t})_{i}}{(q^{k+\alpha};q^{r})_{tj}(q^{s};q^{s})_{i}(q^{t};q^{t})_{n-i}}$$

corresponding properties for the q- Konhauser polynomials and subsequently reduce to corresponding properties for q-Laguerre polynomials for k = 1

SERIES EXPANSION FOR $U_n^{(\alpha)}$ (X; R,P,K,S,T/Q)

$$U_n^{(\alpha)}(x;r,p,k,s,t/q)=\sum_{m=0}^{n}\frac{(q^{\alpha-\beta};q^{s})_{n-m}q^{(\alpha-\beta)m}(q^{r};q^{r})_{m}(q^{s};q^{s})_{n}}{(q^{s};q^{s})_{m}(q^{s};q^{s})_{n-m}(q^{r};q^{r})_{n}}*U_m^{(\beta)}(x;r,p,k,s,t/q)$$

CONNECTION COEFFICIENT

$$D(n,i)=\frac{\left(q^{-nr};q^{r}\right)_{i}\left(q^{1};q^{1}\right)_{i}}{p^{n}\left(q^{k+\alpha};q^{r}\right)_{ti}\left(q^{s};q^{s}\right)_{i}}\sum_{m=0}^{n-i}\frac{\left(q^{-r(n-i)};q^{r}\right)_{m}}{\left(q^{r(1+i)};q^{r}\right)_{m}}*\frac{(q^{t(1+i)};q^{t})_{m}}{\left(q^{t};q^{t}\right)_{m}}q^{-m(ti+\alpha-r+1)}$$

RODRIGUE'S FORMULA

$$u_n^{(\alpha)}(x;r;p;k;s;t/q)-\frac{x^{k+\alpha-s}(-px^{r},q^{r})_{\infty}}{(x^{r};q^{r})_{n}}=x^{-n}\sum_{j=0}^{n}[\frac{(q^{-ns};q^{s})_{j}x^{[(k+\alpha-s)]/r}}{(q^{s};q^{s})_{j}(-px^{r/s};q^{r})_{\infty}}]_{x=k}$$

$$=\frac{x^{k+\alpha-s}(-px^{r},q^{r})_{\infty}}{(x^{r};q^{r})_{n}}D_k^{k}\{\frac{x^{\beta+n}}{(-px^{r},q^{r})_{\infty}}\}]_{x=k}$$

$$where\beta-\frac{k+\alpha-s}{s}$$

which is Rodrigue's formula.

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