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COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE MAPPINGS IN FUZZY METRIC SPACES

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Abstract : We prove common fixed point theorem for weakly compatible mappings in fuzzy metric space. We extend results of Pathak, Khan and Tiwari to fuzzy metric space.

Introduction: In 1965 the concept of fuzzy sets was introduced by Zadeh [14]. It was developed extensively by many authors and used in various fields. Especially, Deng [3], Erceg [4], and Kramosil and Michalek [10] have introduced the concepts of fuzzy metric spaces in different ways.

Recently, George and Veeramani [7],[8] modified the concept of fuzzy metric spaces introduced by kramosil and Michalek [10] and defined the Hausdoff topology of fuzzy metric spaces. They showed also that every metric induces a fuzzy metric.

Grabiec [6] extended the well known fixed point theorem of Banach [1] and Edelstein [5] to fuzzy metric spaces in the sense of Kramosil and Michalek [10].

Here we extend results of Pathak, Khan and Tiwari [12] to fuzzy metric space.

Preliminaries

Definition 1 : [13] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an Abelian topological monoid with the unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all a, b, c, d are in [0,1].

Examples of t-norm are $a * b = ab$ and $a * b = \min \{a,b\}$.

Definition 2 : [10] The 3-tuple (X, M, *) is called a fuzzy metric space (shortly FM-space) if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions for all x, y, z in X and t, s > 0,

(FM-1) $M(x, y, 0) = 0$,

(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$,

(FM-5) $M(x, y, .) : [0,1] \rightarrow [0,1]$ is left continuous.

In what follows, (X, M, *) will denote a fuzzy metric space. Note that M(x, y, t) can be thought as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$ and $M(x, y, t) = 0$ with ∞ and we can find some topological properties and examples of fuzzy metric spaces in (George and Veeramani [7]).

Example 1 : [7] Let (X, d) be a metric space. Define $a * b = ab$ or $a * b = \min\{a, b\}$ and for all x, y in X and $t > 0$,

$$M(x,y,t) = \frac{t}{t + d(x,y)}$$

Then (X, M, *) is a fuzzy metric space. We call this fuzzy metric M induced by the metric d the standard fuzzy metric.

For workers of this line we are giving Lemmas.

Lemma 1 : [6] For all $x,y \in X$, $M(x,y,.)$ is non-decreasing.

Definition 3 : [6] Let (X, M, *) be a fuzzy metric space :

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$), if

$$\lim_{n\to\infty} M(x_n, x, t) = 1,$$

for all $t > 0$.

(2) A sequence $\{x_n\}$ in X called a Cauchy sequence if

$$\lim_{n\to\infty} M(X_{n+p}, x_n, t) = 1,$$

for all $t > 0$ and $p > 0$.

(3) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Remark 1 : Since * is continuous, it follows from (FM-4) that the limit of the sequence in FM-space is uniquely determined.

Let (X,M,*) be a fuzzy metric space with the following condition:

(FM-6) $\lim_{t\to\infty} M(x,y,t) = 1$ for all $x,y \in X$.

Lemma 2 : [11] If for all $x,y \in X$, $t > 0$ and for a number $k \in (0,1)$,

$$M(x,y,kt) \geq M(x,y,t)$$

then $x = y$.

Lemma 3 : [2,11] Let $\{y_n\}$ be a sequence in a fuzzy metric space (X, M, *) with the condition (FM-6). If there exists a number $k \in (0,1)$ such that

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$$

for all $t > 0$ and n = 1, 2, …….. then $\{y_n\}$ is a Cauchy sequence in X.

Definition 4 : [9] A pair of mappings S and T is called weakly compatible pair in fuzzy metric space if they commute at coincidence points; i.e. , if Tu = Su for some $u \in X$, then TSu = STu.

It is easy to see that if S and T are compatible , then they are weakly compatible and the converse is not true in general.

Example 2 : Let $X = R_+$. Define S and T by :

$$Sx = x \text{ and } Tx = 2x-1 ; \quad Sx = Tx \text{ iff } x = 1,$$

As ST(1) = S(1) = 1, TS(1) = T(1) = 1

Therefore {S,T} are weakly compatible.

Let Φ be the set of all continuous and increasing functions $\phi_i :[0,1] \to [0,1]$ in any coordinate and $\phi_i(t) > t$ for all $t \in [0,1)$ and i = 1, 2, 3, 4, 5.

Main Results

We extend results of Pathak, Khan and Tiwari [12] to fuzzy metric spaces.

Theorem 1 : Let (X, M, *) be a complete fuzzy metric space with $t*t \geq t$ for all $t \in [0,1]$. Let A, B, S and T be mappings of X into itself such that

(1.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(1.2) there exists a constant $k \in (0,1)$ such that

$$\begin{aligned} M^{2p}(Ax, By, kt) \geq \min\{ & \phi_1(M^{2p}(Sx, Ty, t)), \phi_2(Mq(Sx, Ax, t).M^{q'}(Ty, By, t)) , \\ & \phi_3 (M^{r}(Sx, By, (2-\alpha) t).M^{r'}(Ty, Ax, t)), \\ & \phi_4(M^{s}(Sx, Ax, t).M^{s'}(Ty, Ax, t)), \\ & \phi_5(M^{l}(Sx, By, (2-\alpha) t). M^{l'}(Ty, By, t)) \}, \end{aligned}$$

for all $x, y \in X$, $\alpha \geq 0$, $\alpha \in (0,2)$, $t > 0$, $\phi_i \in \Phi$, i = 1, 2, 3, 4, 5, $0 < p, q, q', r, r', s, s' l, l' \leq 1$, such that $2p = q + q' = r + r' = s + s' = l + l'$.

(1.3) If the pairs {A, S} and {B, T} are weakly compatible, then

A, B, S and T have a unique common fixed point in X.

Proof : Since $A(X) \subset T(X)$, for arbitrary point x_0 in X, there exists a point

$x_1 \in X$ such that $T x_1 = Ax_0$. Since $B(X) \subset S(X)$, for this point x_1 we can choose a point $x_2 \in X$ such that $Sx_2 = Bx_1$ and so on. Continuing in this manner, we can define a sequence $\{y_n\}$ in X such that

(1.4) $y_{2n} = T x_{2n+1} = Ax_{2n}$ and $y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$ for $n = 1, 2, 3, \ldots,$

We need following Lemma for the proof of our mail Theorem.

Lemma 4 : Let A,B, S and T be self-mappings of a fuzzy metric space (X,M, *) satisfying the conditions (1.1) and (1.2). Then the sequence $\{y_n\}$ denoted by (1.4) is a Cauchy sequence in X.

Proof. For $t > 0$, By putting $x = x_{2n}$ and $y = x_{2n+1}$ in (1.2), $\alpha = 1 - q$, with $q \in (0,1)$ we have

$M^{2p}(y_{2n}, y_{2n+1}, kt) = M^{2p}(Ax_{2n},Bx_{2n+1}, kt)$

$$
\begin{aligned}
M^{2p}(Ax_{2n}, Bx_{2n+1} , kt) \geq \min\{ & \phi_1(M^{2p}(Sx_{2n}, Tx_{2n+1} , t)), \\
& \phi_2(M^q(Sx_{2n}, Ax_{2n}, t).M^{q'}(Tx_{2n+1} , Bx_{2n+1} , t)) , \\
& \phi_3(M^r(Sx_{2n}, Bx_{2n+1},(2-\alpha) t).M^{r'}(Tx_{2n+1}, Ax_{2n}, t)) \\
& \phi_4(M^s(Sx_{2n}, Ax_{2n}, t).M^{s'}(Tx_{2n+1}, Ax_{2n}, t)), \\
& \phi_5(M^l(Sx_{2n}, Bx_{2n+1}, (2-\alpha)t). M^{l'}(Tx_{2n+1}, Bx_{2n+1}, t))\},
\end{aligned}
$$

$$
\begin{aligned}
M^{2p}(y_{2n}, y_{2n+1} , kt) \geq \min\{ & \phi_1(M^{2p}(y_{2n-1}, y_{2n} ,t)),\phi_2(M^q(y_{2n-1}, y_{2n},t).M^{q'}(y_{2n} , y_{2n+1},t)), \\
& \phi_3(M^r(y_{2n-1}, y_{2n+1}, (1+q)t).M^{r'}(y_{2n}, y_{2n}, t)), \\
& \phi_4(M^s(y_{2n-1}, y_{2n}, t).M^{s'}(y_{2n}, y_{2n}, t)), \\
& \phi_5(M^l(y_{2n-1}, y_{2n+1}, (1+q)t). M^{l'}(y_{2n}, y_{2n+1}, t)) \},
\end{aligned}
$$

$$
\begin{aligned}
M^{2p}(y_{2n}, y_{2n+1} , kt) \geq \min \{ & \phi_1(M^{2p}(y_{2n-1}, y_{2n} , t)), \\
& \phi_2(M^q(y_{2n-1}, y_{2n}, t).M^{q'}(y_{2n} , y_{2n+1} , t)) , \\
& \phi_3(M^r(y_{2n-1}, y_{2n}, t)* M^r(y_{2n}, y_{2n+1}, qt))1 \\
& \phi_4 (M^s(y_{2n-1}, y_{2n}, t)1), \\
& \phi_5 (M^l(y_{2n-1}, y_{2n}, t) * M^l(y_{2n}, y_{2n+1}, qt). M^{l'}(y_{2n}, y_{2n+1}, t))\},
\end{aligned}
$$

Since the t-norm * is continuous and M(x, y, .) is continuous , letting $q \to 1$, we have

$$
\begin{aligned}
M^{2p}(y_{2n}, y_{2n+1} , kt) \geq \min \{ & \phi_1(M^{2p}(y_{2n-1}, y_{2n} , t)), \\
& \phi_2(M^q(y_{2n-1}, y_{2n}, t).M^{q'}(y_{2n} , y_{2n+1} , t)),\phi_3(M^r(y_{2n-1}, y_{2n}, t)* M^r(y_{2n}, y_{2n+1}, t)) \\
& \phi_4 (M^s(y_{2n-1}, y_{2n}, t)), \phi_5 (M^l(y_{2n-1}, y_{2n}, t) * M^l(y_{2n}, y_{2n+1}, t). M^{l'}(y_{2n}, y_{2n+1}, t))\},
\end{aligned}
$$

$M(y_{2n}, y_{2n+1} , kt)$

$$(1.5)\quad > \begin{cases} M(y_{2n-1}, y_{2n}, t), & \text{if } M(y_{2n-1}, y_{2n}, t) < M(y_{2n}, y_{2n+1}, t) \\ M(y_{2n}, y_{2n+1}, t), & \text{if } M(y_{2n-1}, y_{2n}, t) \geq M(y_{2n}, y_{2n+1}, t), \end{cases}$$

as $\phi_i(t) > t$ for $0 < t < 1$. Thus $\{M(y_{2n}, y_{2n+1}, t), n \geq 0\}$ is an increasing sequence of positive real numbers in [0,1] and therefore tends to a limit $l \leq 1$. We assert that $l = 1$. If not, $l < 1$ which on letting $n\to\infty$ in (1.5) one gets $l \geq \phi(l) > l$ a contradiction yielding thereby $l = 1$. Therefore for every $n \in N$, using analogous arguments one can show that $\{M(y_{2n+1}, y_{2n+2}, t), n \geq 0\}$ is a sequence of positive real numbers in [0, 1] which tends to a limit $l = 1$. Therefore for every $n \in N$, $M(y_n, y_{n+1}, t) > M(y_{n-1}, y_n, t)$ and

$\lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1$.

Now for any positive integer p

$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * \overset{p\text{-times}}{\ldots} * M(y_{n+p-1}, y_{n+p}, t/p)$.

Since $\lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1$ for $t > 0$, it follows that

$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 * 1 \ldots * 1 = 1$

which shows that $\{y_n\}$ is a Cauchy sequence in X.

Now we prove our main result as follows:

Since X is complete, it follows by Lemma 4, that the sequence $\{y_n\}$ converges to a point z in X. On the other hand, the sub sequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{T x_{2n+1}\}$ of $\{y_n\}$ also converges to the point z .

Now suppose that the subsequence $\{y_{2n}\}$ is contained in S(X) and has a limit in S(X) call it z .

Let $u \in S^{-1}(z)$. Then $Su = z$.

By (1.2) with $\alpha = 1$, we have

$$\begin{aligned} M^{2p}(Au, y_{2n+1}, kt) &= M^{2p}(Au, Bx_{2n+1}, kt) \\ &\geq \min \{ \phi_1(M^{2p}(Su, Tx_{2n+1}, t)), \phi_2(M^q(Su, Au, t).M^{q'}(Tx_{2n+1}, Bx_{2n+1}, t)), \\ &\phi_3(M^r(Su, Bx_{2n+1}, t).M^{r'}(Tx_{2n+1}, Au, t)), \phi_4(M^s(Su, Au, t).M^{s'}(Tx_{2n+1}, Au, t)), \\ &\phi_5(M^l(Su, Bx_{2n+1}, t).\ M^{l'}(Tx_{2n+1}, Bx_{2n+1}, t)) \}, \end{aligned}$$

which implies that as $n \to \infty$, we have

$$M^{2p}(Au, z, kt) \geq \min \{ \phi_2(M^q(z, Au, t)), \phi_3(M^{r'}(z, Au, t)), \phi_4(M^{s+s'}(z, Au, t) \}$$

$$M^{2p}(Au, z, kt) \geq \phi_4(M^{2p}(z, Au, t)) > M^{2p}(z, Au, t)$$

a contradiction. Therefore $Au = z = Su$, i. e. u is a coincidence point of A and S.

Now suppose that the subsequence $\{y_{2n}\}$ is contained in T(X) and has a limit in T(X) call it z .

Let $v \in T^{-1}(z)$. Then $Tv = z$.

Again by (1.2) with $\alpha = 1$, we have

$$M^{2p}(y_{2n}, Bv, kt) = M^{2p}(Ax_{2n}, Bv, kt)$$
$$\geq \min \{\phi_1(M^{2p}(Sx_{2n}, Tv, t)), \phi_2(M^q(Sx_{2n}, Ax_{2n}, t).M^{q'}(Tv, Bv, t)),$$
$$\phi_3(M^r(Sx_{2n}, Bv, t).M^{r'}(Tv, Ax_{2n}, t)), \phi_4(M^s(Sx_{2n}, Ax_{2n}, t).M^{s'}(Tv, Ax_{2n}, t)),$$
$$\phi_5(M^l(Sx_{2n}, Bv, t). M^{l'}(Tv, Bv, t)) \},$$

which implies that as $n \to \infty$, we have

$$M^{2p}(z, Bv, kt) \geq \min \{ \phi_1(M^{2p}(z, z, t)), \phi_2(M^q(z, z, t).M^{q'}(z, Bv, t)),$$
$$\phi_3(M^r(z, Bv, t).M^{r'}(z, z, t)), \phi_4(M^s(z, z, t).M^{s'}(z, z, t)),$$
$$\phi_5(M^l(z, Bv, t). M^{l'}(z, Bv, t)) \},$$

or

$$M^{2p}(z, Bv, kt) \geq \min \{ \phi_1(1), \phi_2(M^{q'}(z, Bv, t)), \phi_3(M^r(z, Bv, t)),$$
$$\phi_4(1), \phi_5(M^{l+l'}(z, Bv, t)) \},$$

or

$$M^{2p}(z, Bv, kt) \geq \phi_5(M^{l+l'}(z, Bv, t)) > M^{2p}(z, Bv, t)$$

a contradiction Therefore $Bv = z$. Since $Tv = z$ thus $Tv = Bv = z$

i.e. v is a coincidence point of B and T.

Since the pair {A,S} is weakly compatible therefore, A and S commute at their coincidence point , i.e. if $ASw = SAw$ or $Az = Sz$.

Similarly, since the pair {B,T} is weakly compatible therefore, B and T commute at their coincidence point , i.e. if $BTw = TBw$ or $Bz = Tz$.

Now, we prove $Az = z$. By (1.2) with $\alpha = 1$, we have

$$M^{2p}(Az, y_{2n+1}, kt) = M^{2p}(Az, Bx_{2n+1}, kt)$$
$$\geq \min \{ \phi_1(M^{2p}(Sz, Tx_{2n+1}, t)), \phi_2(M^q(Sz, Az, t).M^{q'}(Tx_{2n+1}, Bx_{2n+1}, t)),$$
$$\phi_3(M^r(Sz, Bx_{2n+1}, t).M^{r'}(Tx_{2n+1}, Az, t)), \phi_4(M^s(Sz, Az, t).M^{s'}(Tx_{2n+1}, Az, t)),$$
$$\phi_5(M^l(Sz, Bx_{2n+1}, t). M^{l'}(Tx_{2n+1}, Bx_{2n+1}, t)) \},$$

which implies that as $n \to \infty$, we have

$$M^{2p}(Az, z, kt) \geq \min \{ \phi_1(M^{2p}(Az, z, t)), \phi_2(1), \phi_3(M^r(Az, z, t).M^{r'}(z, Az, t)),$$
$$\phi_4(M^{s'}(z, Az, t)), \phi_5(M^l(Az, z, t)) \},$$

$$M^{2p}(Az, z, kt) \geq \phi_1(M^{2p}(Az, z, t) > M^{2p}(Az, z, t)$$

a contradiction. Therefore $Az = z$. Thus $Az = Sz = z$.

Now, we prove $Bz = z$. By (1.2) with $\alpha = 1$, we have

$$M^{2p}(Ax_{2n}, Bz, kt) \geq \min\{ \phi_1(M^{2p}(Sx_{2n}, Tz, t)), \phi_2(M^q(Sx_{2n}, Ax_{2n}, t).M^{q'}(Tz, Bz, t)),$$
$$\phi_3(M^r(Sx_{2n}, Bz, t).M^{r'}(Tz, Ax_{2n}, t)), \phi_4(M^s(Sx_{2n}, Ax_{2n}, t).M^{s'}(Tz, Ax_{2n}, t)),$$
$$\phi_5(M^l(Sx_{2n}, Bz, t).\ M^{l'}(Tz, Bz, t)) \},$$

which implies that as $n \to \infty$, we have

$$M^{2p}(Ax_{2n}, Bz, kt) \geq \min \{ \phi_1(M^{2p}(z, Bz, t)), \phi_2(1), \phi_3(M^{r+r'}(z, Bz, t)),$$
$$\phi_4(M^{s'}(Bz, z, t)), \phi_5(M^l(z, Bz, t)) \},$$

or

$$M^{2p}(z, Bz, kt) \geq \phi_1(M^{2p}(z, Bz, t) > M^{2p}(z, Bz, t) .$$

a contradiction. Therefore $Bz = z$. Since $Tz = z$ thus $Tz = Bz = z$.

Combining the above results, we have

$Az = Bz = Sz = Tz = z$, z is a common fixed point of A, B, S and T.

Fotr the uniqueness of common fixed point let w ($z \neq w$) be another common fixed point of A, B, S and T . Then by (1.2) with $\alpha = 1$, we have

$$M^{2p}(z, w, kt) = M^{2p}(Az, Bw, kt)$$
$$\geq \min \{ \phi_1(M^{2p}(Sz, Tw, t)), \phi_2(M^q(Sz, Az, t).M^{q'}(Tw, Bw, t)),$$
$$\phi_3(M^r(Sz, Bw, t).M^{r'}(Tw, Az, t)), \phi_4(M^s(Sz, Az, t).M^{s'}(Tw, Az, t)),$$
$$\phi_5(M^l(Sz, Bw, t).\ M^{l'}(Tw, Bw, t)) \},$$

or

$$M^{2p}(z, w, kt) \geq \min \{ \phi_1(M^{2p}(z, w, t)), \phi_2(M^q(z, z, t).M^{q'}(w, w, t)),$$
$$\phi_3(M^r(z, w, t).M^{r'}(w, z, t)), \phi_4(M^s(z, z, t).M^{s'}(w, z, t)),$$
$$\phi_5(M^l(z, w, t).\ M^{l'}(w, w, t)) \},$$

$$M^{2p}(z, w, kt) \geq \phi_1(M^{2p}(z, w, t)) > M^{2p}(z, w, t) .$$

a contradiction. Therefore $z = w$.

This completes the proof of the Theorem.

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