Ind Jour Math & Comp Sc. Jhs, Vol. I-II, 2011 Pg. 13 – 22

CERTAIN DUAL SERIES EQUATIONS INVOLVING HAHN POLYNOMIALS IN DISCRETE VARIABLES

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Received on 4/2/2012

ABSTRACT

The present paper deals with dual series equations involving Hahn polynomial in discrete variables.

INTRODUCTION

In past years considerable attention has been drawn of several researchers to the solution of problems involving dual equations involving, for instance, trigonometric series, the Fourier-Bessel series, the Dini series and series of Jacobi and Laguerre polynomials.Many of these problems arise in the investigation of certain classes of mixed boundary value problems in potential theory .For a good account of such problems; one can refer to Sneddon [5]. In particular,dual series equations in which the kernels involve Jacobi polynomials of the same indices were first considered by Noble [4] in 1963. Subsequently Srivastava R.P.[8], Dwivedi[1] ,Thakare [9] also considered dual series equation involving Jacobi polynomials. Srivastava, H.M.[[6],[7]] considered problems of determining the unknown sequence {An} satisfying the general dual series equations

$$(1.1)\qquad \sum_{n=0}^{\infty} A_n \frac{\Gamma(u+n+h+1)}{\Gamma(b+n+h+1)} p_{n+h}^{(a,b)}(x) = f(x); -1 \le x < y,$$

$$(1.2)\qquad \sum_{n=0}^{\infty} A_n \frac{\Gamma(v+n+h+1)}{\Gamma(c+n+h+1)} p_{n+h}^{(c,d)}(x) = g(x); y < x \le 1$$

where h is an arbitrary nonnegative integer ,f(x) and g(x) are prescribed function and in general

min {a,b,c,d,u,v} > -1 .

To solve these equations H.M. Srivastava [6] applied the technique of Noble, called multiplying factor technique with adequate modification.

It is interesting to note that the problems dealt with so far had been those involving continuous variables.

In the present paper we have attempted to deal with a problem involving orthogonal Hanh polynomials in discrete variables defined by relation [3].

$$(1.3)\qquad Q_n(x,a,b,N) = {}_3F_2\left[\begin{matrix} -n, & n+a+b+1, & -x; \\ & a+1, & -N; \end{matrix}\; 1\right]$$

2.STATEMENT OF THE PROBLEMS:

THEOREM I :Let { A_n } be an unknown sequence satisfying the dual series equations:

(2.1) $$\sum_{n=0}^{N-h} A_n\,(u+1)_{n+h} Q_{n+h}\left(x;a,b,N\right) = f\left(x\right); 0 \le x \le y,$$

(2.2) $$\sum_{n=0}^{N-h} A_n \frac{(v+1)_{n+h}\,(c+1)_{n+h}}{(d+1)_{n+h}} Q_{n+h}\left(x;a,b,N\right) = g\left(x\right);$$

y < x ≤N

where h is an arbitrary nonnegative integer,f(x) and g(x) are prescribed functions,

(2.3) a+b = c+d = u+v

(2.4) v > d - k > -1 , u-a+m > 0 and in general

(2.5) min {a,b,c,d,u,v} > -1

then the unknown sequence { A_n } are determined by the relation

(2.6) $$A_n = \frac{(-)^{n+h}(-N)_{n+h}(a+b+1)_{n+h}(a+b+2n+2h+1)}{(n+h)!(a+b+N+2)_{n+h}(v+1)_{n+h}(a+b+1)}$$

$$*\binom{N+a+b+1}{N}^{-1}\Big[\sum_{z=0}^{y}\binom{N-z+v}{N-z}Q_{n+h}(z;u,v,N)F(z)$$

$$+(-)^{k}\sum_{z=y+1}^{N}\binom{z+u}{z}Q_{n+h}(z;u,v,N)G(z)\Big],$$

where

(2.7) $$F(z) = \nabla_z^m\Big[\sum_{x=0}^{Z}\binom{x+a}{x}\binom{z-x+m+u-a-1}{z-a}f(x)\Big],$$

(2.8) $$G(z) = \sum_{x=z}^{N}\binom{x-z+v-d+k-1}{x-z}\Delta_x^k\Big[\binom{N-x+d}{N-x}g(x)\Big],$$

(2.9) $$\nabla_x f(x) = f(x) - f(x-1)$$

(2.10) $$\Delta_x f(x) = f(x+1) - f(x).$$

THEOREM II: Let $\{A_n\}$ be a known sequence satisfying the dual series equations:

(2.11) $$\sum_{n=0}^{N-h} A_n (u+1)_{n+h} Q_{n+h}(x;a,b,N) = \phi(x);\ y < x \le N,$$

(2.12) $$\sum_{n=0}^{N-h} A_n \frac{(v+1)_{n+h}(c+1)_{n+h}}{(d+1)_{n+h}} Q_{n+h}(x;a,b,N) = \psi(x); 0 \le x \le y,$$

where the coefficient $\{A_n\}$ is given by (2.6), h is a nonnegative Integer and in addition to the parametric constraints given by (2.3), (2.4) and (2.5).

Then the unknown functions $\phi(x) and \psi(x)$ are given by:

(2.13) $$\phi(x) = \binom{x+a}{x}^{-1} \nabla_x^r [\sum_{z=0}^{y} \binom{x-z+a-u+r-1}{x-z} F(x)$$
$$+(-)^K \sum_{z=y+1}^{x} \binom{z+u}{x}\binom{N-z+v}{N-z}\binom{x-z+a-u+r-1}{x-z} G(z)]$$
$$-\binom{x-a}{x}^{-1} \nabla_x^r [\binom{x+a+r}{x} \sum_{z=0}^{y} \binom{N-z+v}{N-z} M(x,z) \quad F(z)$$
$$+(-)^k \binom{x+a+r}{x} \sum_{z=y+1}^{N} \binom{z+u}{z} M(x,z) \quad G(z)]$$

where r being a nonnegative integer such that

(2.14) $$a-u+r>0, \qquad b-r>-1$$

(2.15) $$M(x,z)=\sum_{n=0}^{h-1} R_n(x,z)$$

and

(2.16) $$R_n(x,z)=\frac{(-)^n(-N\ )_n(a+b+1)_n(u+1)_n(a+b+2n+1)}{n!\ (a+b+N+2)_n(v+1)_n(a+b+1)}$$

$$*\binom{N+a+b+1}{N}^{-1}\ Q_n(x,a+r,b-r,N)\,Q_n(z,u,v,N)$$

and

(2.17) $$\psi(x)=\binom{N-x+d}{N-x}^{-1}(-\Delta_x)^s\Big[\sum_{z=x}^{y}\ \binom{N-z+v}{N-z}\binom{Z+u}{z}^{-1}$$

$$\binom{Z-x+u-c+s-1}{z-x}F(z)+(-)^k\sum_{Z=y}^{N}\binom{Z-x+u-c+s-1}{z-x}G(z)\Big]$$

$$-\binom{N-x+d}{N-x}^{-1}(-\Delta_x)^s\Big[\binom{N-x+d+s}{N-x}\sum_{z=0}^{y}\ \binom{N-z+v}{N-z}k(x,z)F(z)$$

$$+(-)^K\binom{N-x+d+s}{N-x}\sum_{z=y}^{N}\ \binom{Z+u}{z}K(x,z)G(z)\Big].$$

where s is a non-negative integer such that

(2.18) $U > c - s > -1,$

(2.19) $$K(x,z)=\sum_{n=0}^{h-1}\ s_n(x,z)$$

(2.20) $$s_n(x,z)\ =\ \frac{(-)^n(-N)_n(c+d+1)_n(c-s+1)_n(c+d+2n+1)}{n!(c+d+N+2)_n(d+s+1)_n(c+d+1)}$$

$$*\binom{N+a+b+1}{N}^{-1}Q_n(x;c-s,d+s,N)\,Q_n(z;u,v,N)$$

3.PRELIMINARY RESULT: For the solution of our problems of section (2) the following result involving Hahn polynomials will be required.

(I) The following convenient forms of the orthogonality properties of the Hahn polynomials given by karlin and McGregor (3)

$$(3.1)\quad \sum_{X=0}^{N} \binom{X+a}{x}\binom{N-x+b}{N-x} Q_n(x,a,b,N)Q_m(x,a,b,N)$$

$$= \frac{(-)^n n!(N+a+b+2)_n (b+1)_n (a+b+1)}{(-N)_n (a+1)_n (a+b+1)_n (2n+a+b+1)} \binom{N+a+b+1}{N} \delta_{mn}$$

$$where 0 \le m,n \le n$$

and

$$(3.2)\quad \sum_{n=0}^{N} \frac{(-)^n (-N)_n (a+1)_n (a+b+1)_n (2n+a+b+1)}{n!(N+a+b+2)_n (b+1)_n (a+b+1)} * Q_n(x,a,b,N) \quad Q_m(Y,a,b,n)$$

$$= \binom{x+a}{x}^{-1} \binom{N-x+b}{N-x}^{-1} \binom{N+a+b+1}{N} \delta_{xy}$$

where x, y are integers and $o \le x, y \le N$.

(II) For p > 0 and a> -1 we have

$$(3.3)\quad \sum_{x=0}^{z} \binom{x+a}{x}\binom{z-x+p-1}{z-x} Q_n(x;a,b,N)$$

$$= \binom{z+a+p}{z} Q_n(z;a+p,b-p,N)$$

which is the slightly modified form of the following summation formula given by Gasper [2].

$$Q_n(z;a+p,b-p,N) = \sum_{x=o}^{Z} \binom{z}{x} \frac{(a+1)_x (p)_{z-x}}{(a+p+1)_z} Q_n(x;a,b,N).$$

by use of transformation we can write equation (1.3) as

$$(3.4)\quad Q_n(x;a,b,N) = \frac{(-1)^n (b+1)_n}{(a+1)_n} {}_3F_2\left[\begin{pmatrix} -n, & n+a+b+1, & -N+x; & \\ & & & 1 \\ & b+1, & -N; & \end{pmatrix}\right]$$

hence we get the following relation

$$(3.5)\quad Q_n(x;a,b,N) = \frac{(-1)^n (b+1)_n}{(a+1)_n} Q_n(N-x;b,a,N)$$

On replacing a, b, x and z respectively by b, a, N-x and N-z and using the relation (3.5) in equation (3.3) gives its following complementary result.

$$(3.6)\qquad \sum_{x=z}^{N}\binom{N-x+b}{N-x}\binom{x-z-+p-1}{x-z}\; Q_n(x;a,b,N)$$
$$=\frac{(a-p+1)_n\ (b+1)_n}{(b+p+1)_n\ (a+1)_n}\ \binom{N-z+b+p}{N-z}Q_n(z;a-p,b+p,N)$$

for p > 0 and b > -1.

(III) In our analysis we shall also use the following two difference formulae involving Hahn polynomials.

$$(3.7)\qquad \nabla_x^m\left[\binom{x+a+m}{x}Q_n(x,a+m,b-m,N)\right]=\binom{x+a}{x}\quad Q_n(x;a,b,N)$$

for nonnegative integer m and a > -1
and

$$(3.8)\qquad \Delta_x^m\left[\binom{N-x+b+m}{N}Q_n(x;a-m,b+m,N)\right]$$
$$=\frac{(-)^m\,(b+m+1)_n(a+1)_n}{(a-m+1)_n(b+1)_n}\binom{N-x+b}{N-x}\quad Q_n(x;a,b,N)$$

for b > -1 and integer m ≥ 0.
From equations (1.3) and (3.4) we see that the above results are special cases of,

$$(3.9)\qquad \nabla_x^m\left[\binom{x-u+a+m}{x-u}{}_{E+1}F_{G+1}\left[\begin{matrix}-x+u, & (e); \\ a+1+m, & (g);\end{matrix}\ t\right]\right]$$
$$=\binom{x-u+a}{x-u}{}_{E+1}F_{G+1}\left[\begin{matrix}-x+u, & (e) \\ a+1, & (g);\end{matrix}\ t\right]$$

(when u=0, e=2, g=1, e_1=n+a+b+1, e_2= -n, g_1 =-N and t=1)
and

$$(3.10)\qquad \Delta_x^m[\binom{u-x+b+m}{u-x}{}_{E+1}F_{G+1}\left[\begin{matrix}-u+x, & (e); \\ b+1+m, & (g),\end{matrix}\ t\right]]$$
$$=(-)^m\binom{u-x+b}{u-x}{}_{E+1}F_{G+1}\left[\begin{matrix}-u+x, & (e); \\ b+1, & (g),\end{matrix}\ t\right]]$$

(when u=N , E=2 ,G=1 , e_1= n+a+b+1 , e_2 = -n , g_1= -N and t=1) respectively.

To prove (3.9) consider

$$\nabla_x[\binom{x-u+a+m}{x-u}{}_{E+1}F_{G+1}\left[\begin{matrix}-x+u, & (e); \\ a+1+m, & (g),\end{matrix}\ t\right]]$$

$$=\nabla_x\Big[\sum_{r=0}^{\infty}\frac{\Gamma(x-u+a+1+m)}{\Gamma(a+1+m+r)\Gamma(x-u+1)}.\frac{(-x+u)_r(e)_r}{(g)_r r!}t^r\Big]$$

$$=\sum_{r=0}^{\infty}\frac{(e)_r}{\Gamma(a+1+m+r)((g))_r r!}t^r$$

$$*\Big[\frac{\Gamma(x-u+a+1+m)(-x+u)_r}{\Gamma(x-u+1)}-\frac{\Gamma(-x+u+a+m)(-x+u+1)_r}{\Gamma(x-u)}$$

$$=\sum_{r=0}^{\infty}\frac{\Gamma(x-u+a+m)}{\Gamma(a+m+r)\Gamma(x-u+1)}.\frac{(-x+u)_r((e))_r}{((g))_r r!}\quad t^r$$

$$=\binom{x-u+a+m-1}{x-u}{}_{E+1}F_{G+1}\left[\begin{matrix}-x+u, & (e); \\ a+m, & (g);\end{matrix}\; t\right].$$

hence by iteration we get the required result (3.9) in a similar manner we can easily obtain (3.10).

4. PROOF OF THE THEOREM I: To our series equation (2.1) multiply with $\binom{x+a}{x}\binom{z-x+m+p-1}{z-x}$, for some suitable p and non-negative integer m, then summing the series from x=0 to x=z on both sides, and using the summation formula (3.3), we get

$$\text{(4.1)}\qquad \sum_{n=0}^{N-h}A_n(u+1)_{n+h}\binom{z+a+m+p}{z}Q_{n+h}(z;a+m+p,b-m-p,N)$$
$$=\sum_{x=0}^{z}\binom{x+a}{x}\binom{z-x+m+p-1}{z-x}f(x),$$

where $0 \le z \le y$, $a > -1$ and $p+m > 0$.

On operation both sides of this last equation (4.1) by ∇_z, m times and using the difference formula (3.7), we have

$$\text{(4.2)}\qquad \sum_{n=0}^{N-h}A_n(u+1)_{n+h}\binom{z+a+p}{z}Q_{n+h}(z;a+p,b-p,N)$$
$$=\nabla_z^m\Big[\sum_{x=0}^{z}\binom{x+a}{x}\binom{z-x+m+p-1}{z-x}f(x)\Big]$$

where $0 \le z \le y$, $a > -1$ $p+m > 0$ and $a+p > -1$.

Next we multiply our series equation (2.2) by $\binom{N-x+d}{N-x}$ and operate both sides by Δ_x, k times for a non – negative integer k, using the difference formula (3.8), we thus obtain

$$(4.3)\quad \sum_{n=0}^{N-h} A_n \frac{(v+1)_{n+h}(c+k+1)_{n+h}}{(d-k+1)_{n+h}} \binom{N-x+d-k}{N-x} Q_{n+h}(x;c+k,d-k,N)$$
$$= (-)^k \Delta_x^k \left[\binom{N-x+d}{N-x} g(x)\right],$$

where $y < x \le N$ and $d-k > -1$.

For a suitable constant q , multiply equation (4.3) by $\binom{x-z+q+k-1}{x-z}$ then summing the series from x=z to x=N on both sides, using the summation formula (3.6) we get

$$(4.4)\quad \sum_{n=0}^{N-h} A_n (v+1)_{n+h} \frac{(c-q+1)_{n+h}}{(d+q+1)_{n+h}} \binom{N-z+d+q}{N-z} * Q_{n+h}(z;c-q,d+q,N)$$

$$= \sum_{x=z}^{N} \binom{x-z+q+k-1}{x-z} (-\Delta_x)^k \left[\binom{N-x+d}{N-x} g(x)\right].$$

where $y < z \le N$, $d-k > -1$, $q+k > 0$ and integer $k \ge 0$.

Now under the parametric condition (2.3) If we choose p and q such that

$a+p = c-q = u$ and $b-p = d+q = v$.

then equations (4.2) and (4.4) can be written as

$$(4.5)\qquad \sum_{n=0}^{N-h} A_n (u+1)_{n+h} \binom{z+u}{z} Q(z;u,v,N) = F(z)$$

where $0 \le z \le y, a > -1, \ u > -1$ *a*nd F(z)is given by (2.7)and

$$(4.6)\qquad \sum_{n=0}^{N-h} A_n (u+1)_{n+h} \binom{N-z+v}{N-z} Q_{n+h}(z;u,v,N) = (-)^k G(z),$$

where $y < z \le N, d-k > -1, v > -1$ and G(z)is given by (2.8)

Now , multiplying equation (4.5) by $\binom{N-z+v}{N-z}$ $Q_j(z;u,v,N)$ and summing the series from z=0 to y on both sides , we have

$$\sum_{n=0}^{N-h} A_n (u+1)_{n+h} \sum_{z=0}^{y} \binom{z+u}{z} \binom{N-z+v}{N-z} * Q_{n+h}(z;u,v,n) \quad Q_j(z;u,v,N)$$

(4.7)

$$= \sum_{z=0}^{y} \binom{N-z+n}{N-z} \quad Q_j(z;u,v,N) \quad F(z)$$

next if we multiply equation (4.6)by $\binom{z+u}{z}$ Q_j (z;u,v,N) and summing the from z = y+1 to N , we get

$$(4.8) \qquad \sum_{n=0}^{N-h} A_n (u+1)_{n+h} \sum_{z=y+1}^{N} \binom{z+u}{z} \binom{N-z+v}{N-z} * Q_{n+h}(z;u,v,N) Q_j(z;u,v,N) G(z).$$

$$= (-)^k \sum_{z=y+1}^{N} \binom{z+u}{z} Q_j(z;u,v,N) G(z).$$

On adding (4.7) and (4.8) , we obtain

$$(4.9) \qquad \sum_{n=0}^{N-h} A_n (u+1)_{n+h} \sum_{z=0}^{N} \binom{z+u}{z} \binom{N-z+v}{N-z} * Q_{n+h}(z;u,v,N) Q_j(z;u,v,N)$$

$$= \sum_{z=0}^{y} \binom{N-z+v}{N-z} Q_j(z;u,v,N) F(z) \quad + (-)^k \sum_{z=y+1}^{N} \binom{z+u}{z} Q_j(z;u,v,N) G(z)$$

which with the help of orthogonality property (3.1), gives required result (2.6) under the parametric conditions (2.3) , (2.4) and (2.5).

Hence Theorem- I is proved.

5. **THEOREM II**. For non-negative integer`r` we can easily change equation (2.11) , by applying the difference formula (3.7) into the form

$$(5.1) \qquad \phi(x) = \binom{x+a}{x}^{-1} \nabla_x^r \left[\binom{x+a+r}{x} * \sum_{n=0}^{N-h} A_n (u+1)_{n+h} Q_{n+h}(x; a+r, b-r, N) \right],$$

where $y < x \leq N$.

By method applied for theorem I,the theorem II can be proved.

REMARK: - From equation (1.3), it can be easily verified that

$$\lim_{N\to\infty} Q_n(xN; a,b,N) = \frac{n!}{(1+a)_n} P_n^{(a,b)}(1-2x)$$

and

$$\lim_{N\to\infty} Q_n(x; a-1, (1-b)N/c, N) = M_n(x;a,b)$$

where M_n (x;a,b) is called Mexiner polynomials defined as see [9]

$$M_n(x;b,c) = {}_2F_1(-n, -x; b; 1-c^{-1})$$

and the orthogonality property relation of Mexiner polynomials is

$$\sum_{x=0}^{\infty} M_n(x;a,b) M_m(x;a,b) \frac{(a)_x}{x!} b^x = 0$$

$$For\ m \neq n; \qquad 0 < b < 1; a > 0$$

hence our dual series equations (2.1) and (2.2), on replacing x by (1-x)N/2 , taking $\mathrm{Lim} N \to \infty$ and adjusting the parameters reduces into srivastava equation (1.1) and (1.2) respectively involving Jacobi polynomials of continuous variables.

Further, if we replace a, b,c,d respectively by a-1 , (1-b)N /b, c-1 , (1-d)N /d and taking Lim N–›∞ in (2.1) and (2.2), we get the following dual series equations involving Mexiner polynomials

$$\sum_{n=0}^{\infty} A_n (u+1)_{n+h} M_{n+h}(x;a,b) = F(x); \qquad 0 \le x \le y$$

and

$$\sum_{n=0}^{\infty} A_n (\frac{dv}{1-d})^{n+h} (c)_{n+h} M_{n+h}(x;c,d) = g(x); \qquad y < x \le \infty$$

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